1 Divisibility Induction

Prove that for all $n \in \mathbb{N}$ with $n \geq 1$, the number $n^3 - n$ is divisible by 3. (**Hint**: recall the binomial expansion $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$)

2 Make It Stronger

Let $x \geq 1$ be a real number. Use induction to prove that for all positive integers n, all of the entries in the matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^n$$

are $\leq xn$. (Hint 1: Find a way to strengthen the inductive hypothesis! Hint 2: Try writing out the first few powers.)

3 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_k \in \{0, 1\}$.

4 Division Algorithm

Let $a, b \in \mathbb{Z}$, $b \neq 0$. In this problem, we will prove, using the WOP, that there exists unique integers q, r such that $0 \leq r < |b|$ and $a = qb + r$. Here, q is called the *quotient* and r is called the *remainder*.

(a) Let $A = \{a - qb | q \in \mathbb{Z} \wedge a - qb \geq 0\}$. Show that A is non-empty (keep in mind that we must consider the case where a is negative)

(b) Use the WOP to show that there exists $q, r \in \mathbb{Z}$ such that $a = qb + r$, and $0 \leq r < r$.

(c) Show that the q and r from part b are unique