

1 Unions and Intersections

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

- (a) $A \cap B$, where A is countable, and B is uncountable
- (b) $A \cup B$, where A is countable, and B is uncountable
- (c) $\bigcap_{i \in A} S_i$ where A is a countable set of indices and each S_i is an uncountable set.

2 Counting Cartesian Products

For two sets A and B , define the cartesian product as $A \times B = \{(a, b) : a \in A, b \in B\}$.

- (a) Given two countable sets A and B , prove that $A \times B$ is countable.
- (b) Given a finite number of countable sets A_1, A_2, \dots, A_n , prove that

$$A_1 \times A_2 \times \dots \times A_n$$

is countable.

- (c) Consider an infinite number of countable sets: B_1, B_2, \dots . Under what condition(s) is $B_1 \times B_2 \times \dots$ countable? Prove that if this condition is violated, $B_1 \times B_2 \times \dots$ is uncountable.

3 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

- (a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

- (b) You want to determine whether a program P prints "Hello World!" before running the k th line in the program. Is there a computer program that can perform this task? Justify your answer.
- (c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.