

Due: Sunday, 06/28 at 10:00 PM  
Grace period until Sunday 06/28 at 11:59 PM

## 1 Propositional Practice

Convert the following English sentences into propositional logic and the following propositions into English. State whether or not each statement is true with brief justification.

- (a) There is a real number which is not rational.
- (b) All integers are natural numbers or are negative, but not both.
- (c) If a natural number is divisible by 6, it is divisible by 2 or it is divisible by 3.
- (d)  $(\forall x \in \mathbb{Z}) (x \in \mathbb{Q})$
- (e)  $(\forall x \in \mathbb{Z}) (((2 \mid x) \vee (3 \mid x)) \implies (6 \mid x))$
- (f)  $(\forall x \in \mathbb{N}) ((x > 7) \implies ((\exists a, b \in \mathbb{N}) (a + b = x)))$

## 2 Proof Practice

- (a) Prove that  $\forall n \in \mathbb{N}$ , if  $n$  is odd, then  $n^2 + 1$  is even.
- (b) Prove that  $\forall x, y \in \mathbb{R}$ ,  $\min(x, y) = (x + y - |x - y|)/2$ .
- (c) Prove that  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ .
- (d) Suppose  $A \subseteq B$ . Prove  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ .

## 3 Open Set Intersection

For  $a, b \in \mathbb{R}$ , define an open interval  $(a, b)$  as the set  $\{x \in \mathbb{R} \mid x > a \wedge x < b\}$

- (a) By this definition, is the empty set an open interval?
- (b) Let  $(a, b)$  and  $(c, d)$  be two open intervals. Prove that  $(a, b) \cap (c, d)$  is an open interval
- (c) Let  $I_1, \dots, I_k$  be a finite sequence of open intervals. Prove that for every  $k \in \mathbb{N}$ ,  $I_1 \cap I_2 \cap \dots \cap I_k$  is an open interval

- (d) Prove that a set containing exactly one number is not an open interval. (**Hint:** You may use the fact that between any two real numbers, there is another real number)
- (e) Let  $I_1, I_2, \dots$  be an infinite sequence of open intervals. Is it always true that  $\bigcap_{k=1}^{\infty} I_k$  is an open interval? (**hint:** Consider the set  $\bigcap_{k=1}^{\infty} (-1/k, 1/k)$ )
- (f) why can we not use induction to prove part e?

## 4 Induction

Prove the following using induction:

- (a) For all natural numbers  $n > 2$ ,  $2^n > 2n + 1$ .
- (b) For all positive integers  $n$ ,  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ .
- (c) For all positive natural numbers  $n$ ,  $\frac{5}{4} \cdot 8^n + 3^{3n-1}$  is divisible by 19.

## 5 Make It Stronger

Suppose that the sequence  $a_1, a_2, \dots$  is defined by  $a_1 = 1$  and  $a_{n+1} = 3a_n^2$  for  $n \geq 1$ . We want to prove that

$$a_n \leq 3^{2^n}$$

for every positive integer  $n$ .

- (a) Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply  $a_n \leq 3^{2^n}$ ? Show why this does not work.
- (b) Try to instead prove the statement  $a_n \leq 3^{2^n - 1}$  using induction. Does this statement imply what you tried to prove in the previous part?

## 6 A Coin Game

(10 Points) Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of  $n$  coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be  $3 \cdot 2 = 6$ ). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of  $n$  coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be  $\frac{n(n-1)}{2}$ . (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

## 7 Preserving Set Operations

For a function  $f$ , define the image of a set  $X$  to be the set  $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$ . Define the inverse image or preimage of a set  $Y$  to be the set  $f^{-1}(Y) = \{x \mid f(x) \in Y\}$ . Prove the following statements, in which  $A$  and  $B$  are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

*Recall: For sets  $X$  and  $Y$ ,  $X = Y$  if and only if  $X \subseteq Y$  and  $Y \subseteq X$ . To prove that  $X \subseteq Y$ , it is sufficient to show that  $(\forall x) ((x \in X) \implies (x \in Y))$ .*

(a)  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ .

(b)  $f(A \cup B) = f(A) \cup f(B)$ .

## 8 Bijective Or Not

For each of the following, determine whether or not it's an injection, surjection, and bijection. Please prove your claims.

(a)  $f(x) = 10^{-5}x$

(i) for domain  $\mathbb{R}$  and range  $\mathbb{R}$

(ii) for domain  $\mathbb{Z} \cup \{\pi\}$  and range  $\mathbb{R}$

(b)  $f(x) = \{x\}$ , where the domain is  $D = \{0, \dots, n\}$  and the range is  $\mathcal{P}(D)$ , the powerset of  $D$  (that is, the set of all subsets of  $D$ ).

(c) Consider the number  $X = 1234567890$ , and obtain  $X'$  by shuffling the order of the digits of  $X$  (for example, 2134756890). Is  $f(i) = (i + 1)^{\text{st}} \text{ digit of } X'$  a bijection from  $\{0, \dots, 9\}$  to itself?

## 9 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

- 1. What sources (if any) did you use as you worked through the homework?**
- 2. If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
- 3. How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
- 4. Roughly how many total hours did you work on this homework?**