Due: Sunday, July 19, 2020 at 10:00 PM
Grace period until Sunday, July 19, 2020 at 11:59 PM

1 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

(a) We place p beads onto a string. How many different ways are there construct such a sequence of p beads with up to k different colors?

(b) How many sequences of p beads on the string are there that use at least two colors?

(c) Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p.)

[*Hint*: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

(d) Use your answer to part (c) to prove Fermat's little theorem.

2 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 7-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

i. How many different 13-card bridge hands are there? ii. How many different 13-card bridge hands are there that contain no aces? iii. How many different 13-card bridge hands are there that contain all four aces? iv. How many different 13-card bridge hands are there that contain exactly 6 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros?

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.
i. How many different anagrams of ALABAMA are there? ii. How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if: (1) C is the left neighbor of E; (2) C is on the left of E (and not necessarily E's neighbor)

(h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

(j) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

(l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?

(m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?

(n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

3 Good Khalil Hunting

As a sidejob, Khalil is also working as a janitor in Berkeley EECS. One day, he notices a problem on the board and decides to solve it. The problem is as follows: **Find all homeomorphically irreducible trees having 10 vertices.** A tree is homeomorphically irreducible if it has no vertices of degree 2. Assume all vertices and edges are indistinguishable from another.

Let's help Khalil solve this using strategic **casework**. We will partition the problem based off the number of the leaves in the tree. For sake of clarity, label the vertices $v_1, \ldots, v_{10}$ and their degrees $d_1, \ldots, d_{10}$ in decreasing order of degree.

(a) Show that the number of leaves, ℓ, we can have is $6 \leq \ell \leq 9$. (*Hint*: What do you know about the degrees of a leaf? What about a non-leaf in this case?)

For the following parts, drawings are neither necessary nor sufficient for your answer, but are highly encouraged to help you get the answer. Please briefly justify your answers by formulating equations involving the degrees of the vertices, along with short explanations.

(b) How many 10 vertex, homeomorphically irreducible trees of 9 leaves are there? Justify your answer.

(c) How many 10 vertex, homeomorphically irreducible trees of 8 leaves are there? Justify your answer.

(d) How many 10 vertex, homeomorphically irreducible trees of 7 leaves are there? Justify your answer.

(e) How many 10 vertex, homeomorphically irreducible trees of 6 leaves are there? Justify your answer.

In total, you should have counted 10 trees. Great work!

4 August Absurdity

Since March Madness was cancelled, the council unanimously decided to have August Absurdity instead - an online Discrete Mathematics tournament! There are 64 teams (including Cal) in the single-elimination tournament - that means, every match is between two teams and will decide a winner who moves on to the next round and a loser who is eliminated from the tournament. Thus the first round will have 64 teams, the next will have 32, and so on until 1 remains. There is a single, randomly initialized, starting bracket.

(a) How many tournament outcomes exist such that Cal wins the entire tournament?

(b) In the first round, Cal will face a no-name school called LJSU (some people call it Stanfurd?). The format of each match is as follows: Each of the two teams have 8 players labelled from 1 to 8. They play a series of games. In the first game, the two 1's play each other. The loser of the game is eliminated and replaced by the next player of the same team until all players from one team are eliminated, ending the match. What is the number of possible sequences of games such that Cal wins the match?

(c) Cal employs a blasphemous strategy that even baffles themselves. They place their players in an order such that each player is either taller than all the preceding players or shorter than all the preceding players. Let 1-8 represent the players' heights. An example of a valid ordering: 4, 5, 6, 3, 2, 7, 1, 8. An example of an invalid ordering: 1, 2, 3, 4, 5, 6, 8, 7. (invalid since 7 is neither taller or shorter than all the preceding players). How many such orderings exist?

(d) To keep viewership up after the tournament finishes, the council plans an All-Star match. The 16 greatest players in the league were chosen, including Oski and a tree..? Oski refuses to play on the same team as the tree. How many ways can the 16 players be distributed into two teams of 8 players such that Oski and the tree are in opposite teams?

(e) Provide an explanation for the following combinatorial identity. Hint: Solve the previous part using another method. Those two methods should correspond to the two sides of the equality.

$$\binom{n}{r} - \binom{n-2}{r-2} - \binom{n-2}{r} = 2\binom{n-2}{r-1}$$

5 School Carpool

(a) n males and n females apply to EECS within UC Berkeley. The EECS department only has n seats available. In how many ways can it admit students? Use the above story for a combinatorial argument to prove the following identity:

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

(b) Among the n admitted students, there is at least one male and at least one female. On the first day, the admitted students decide to carpool to school. The male(s) get in one car, and the female(s) get in another car. Use the above story for a combinatorial argument to prove the following identity:

$$\sum_{k=1}^{n-1} k \cdot (n-k) \cdot \binom{n}{k}^2 = n^2 \cdot \binom{2n-2}{n-2}$$

(Hint: Consider the ways that students are admitted. Also, each car has a driver!)

6 Flippin' Coins

Suppose we have an unbiased coin, with outcomes H and T, with probability of heads $\mathbb{P}[H] = 1/2$ and probability of tails also $\mathbb{P}[T] = 1/2$. Suppose we perform an experiment in which we toss the coin 3 times. An outcome of this experiment is (X_1, X_2, X_3), where $X_i \in \{H, T\}$.

(a) What is the *sample space* for our experiment?

(b) Which of the following are examples of *events*? Select all that apply.

- $\{(H,H,T),(H,H),(T)\}$

- $\{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$
- $\{(T,T,T)\}$
- $\{(T,T,T),(H,H,H)\}$
- $\{(T,H,T),(H,H,T)\}$

(c) What is the complement of the event $\{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}$?

(d) Let A be the event that our outcome has 0 heads. Let B be the event that our outcome has exactly 2 heads. What is $A \cup B$?

(e) What is the probability of the outcome (H,H,T)?

(f) What is the probability of the event that our outcome has exactly two heads?

7 Past Probabilified

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}(\omega)$.

iii. Compute $\mathbb{P}(E_1)$ given event E_1.

iv. Compute $\mathbb{P}(E_2)$ given event E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0,\ldots,p-1\}$ (assume we have two $\{0,\ldots,p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $(\bmod p)$ space.
$E_1 =$ The resulting product is 0.
$E_2 =$ The product is $(p-1)/2$.

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).
$E_1 =$ The graph is complete.
$E_2 =$ vertex v_1 has degree d.

8 Unlikely Events

(a) Toss a fair coin x times. What is the probability that you never get heads?

(b) Roll a fair die x times. What is the probability that you never roll a six?

(c) Suppose your weekly local lottery has a winning chance of $1/10^6$. You buy lottery from them for x weeks in a row. What is the probability that you never win?

(d) How large must x be so that you get a head with probability at least 0.9? Roll a 6 with probability at least 0.9? Win the lottery with probability at least 0.9?

9 Probability Practice

(a) If we put 5 math, 6 biology, 8 engineering, and 3 physics books on a bookshelf at random, what is the probability that all the math books are together?

(b) A message source M of a digital communication system outputs a word of length 8 characters, with the characters drawn from the ternary alphabet $\{0,1,2\}$, and all such words are equally probable. What is the probability that M produces a word that looks like a byte (*i.e.*, no appearance of '2')?

(c) If five numbers are selected at random from the set $\{1,2,3,\ldots,20\}$, what is the probability that their minimum is larger than 5? (A number can be chosen more than once, and the order in which you select the numbers matters)

10 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

1. **What sources (if any) did you use as you worked through the homework?**
2. **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)
3. **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)
4. **Roughly how many total hours did you work on this homework?**