EECS 70 Discrete Mathematics and Probability Theory
Summer 2020 Note 5

# 1 Cardinality: Infinity and Countability

In this note we'll first discuss the question of when two sets have the same cardinality, or size. This is a simple issue for finite sets, but for infinite sets it becomes subtle. We'll see how to formulate the question precisely, and then see several quite surprising consequences. To set the scene, we begin with the simple case of finite sets.

## 1.1 Cardinality

How can we determine whether two sets have the same *cardinality* (or "size")? The answer to this question, reassuringly, lies in early grade school memories: by demonstrating a *pairing* between elements of the two sets. More formally, we need to demonstrate a *bijection* $f$ between the two sets. The bijection sets up a one-to-one correspondence, or pairing, between elements of the two sets. We've seen above how this works for finite sets. In the rest of this lecture, we will see what it tells us about *infinite* sets.

Our first question about infinite sets is the following: Are there more natural numbers $\mathbb{N}$ than there are positive integers $\mathbb{Z}^+$? It is tempting to answer yes, since every positive integer is also a natural number, but the natural numbers have one extra element $0 \notin \mathbb{Z}^+$. Upon more careful observation, though, we see that we can define a mapping between the natural numbers and the positive integers as follows:

$$\begin{array}{lcccccccc} \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ f\downarrow & \searrow & \searrow & \searrow & \searrow & \searrow & \searrow & \\ \mathbb{Z}^+ & 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

Why is this mapping a bijection? Clearly, the function $f : \mathbb{N} \to \mathbb{Z}^+$ is onto because every positive integer is hit. And it is also one-to-one because no two natural numbers have the same image. (The image of $n$ is $f(n) = n+1$, so if $f(n) = f(m)$ then we must have $n = m$.) Since we have shown a bijection between $\mathbb{N}$ and $\mathbb{Z}^+$, this tells us that there are exactly as many natural numbers as there are positive integers! (Very) informally, we have proved that "$\infty + 1 = \infty$."

What about the set of *even* natural numbers $2\mathbb{N} = \{0, 2, 4, 6, ...\}$? In the previous example the difference was just one element. But in this example, there seem to be twice as many natural numbers as there are even natural numbers. Surely, the cardinality of $\mathbb{N}$ must be larger than that of $2\mathbb{N}$ since $\mathbb{N}$ contains all of the odd natural numbers as well? Though it might seem to be a more difficult task, let us attempt to find a bijection between the two sets using the following mapping:

$$\begin{array}{cccccccc} \mathbb{N} & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ f\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 2\mathbb{N} & 0 & 2 & 4 & 6 & 8 & 10 & \dots \end{array}$$

The mapping in this example is also a bijection. $f$ is clearly one-to-one, since distinct natural numbers get mapped to distinct even natural numbers (because $f(n) = 2n$). $f$ is also onto, since every $n$ in the range is hit: its pre-image is $\frac{n}{2}$. Since we have found a bijection between these two sets, this tells us that in fact $\mathbb{N}$ and $2\mathbb{N}$ have the same cardinality!

What about the set of all integers, $\mathbb{Z}$? At first glance, it may seem obvious that the set of integers is larger than the set of natural numbers, since it includes infinitely many negative numbers. However, as it turns out, it is possible to find a bijection between the two sets, meaning that the two sets have the same size! Consider the following mapping $f$:

$$0 \to 0,\ 1 \to -1,\ 2 \to 1,\ 3 \to -2,\ 4 \to 2,\ \dots,\ 124 \to 62,\ \dots$$

In other words, our function is defined as follows:

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even} \\ \frac{-(x+1)}{2}, & \text{if } x \text{ is odd} \end{cases}$$

We will prove that this function $f : \mathbb{N} \to \mathbb{Z}$ is a bijection, by first showing that it is one-to-one and then showing that it is onto.

**Proof (one-to-one):** Suppose $f(x) = f(y)$. Then they both must have the same sign. Therefore either $f(x) = \frac{x}{2}$ and $f(y) = \frac{y}{2}$, or $f(x) = \frac{-(x+1)}{2}$ and $f(y) = \frac{-(y+1)}{2}$. In the first case, $f(x) = f(y) \Rightarrow \frac{x}{2} = \frac{y}{2} \Rightarrow x = y$. Hence $x = y$. In the second case, $f(x) = f(y) \Rightarrow \frac{-(x+1)}{2} = \frac{-(y+1)}{2} \Rightarrow x = y$. So in both cases $f(x) = f(y) \Rightarrow x = y$, so $f$ is injective.

**Proof (onto):** If $y \in \mathbb{Z}$ is non-negative, then $f(2y) = y$. Therefore, $y$ has a pre-image. If $y$ is negative, then $f(-(2y+1)) = y$. Therefore, $y$ has a pre-image. Thus every $y \in \mathbb{Z}$ has a preimage, so $f$ is onto.

Since $f$ is a bijection, this tells us that $\mathbb{N}$ and $\mathbb{Z}$ have the same size.

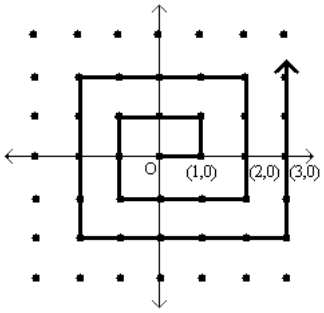
Now for an important definition. We say that a set $S$ is **countable** if there is a bijection between $S$ and $\mathbb{N}$ or some subset of $\mathbb{N}$. Thus any finite set $S$ is countable (since there is a bijection between $S$ and the subset $\{0, 1, 2, \dots, m-1\}$, where $m = |S|$ is the size of $S$). And we have already seen three examples of countable infinite sets: $\mathbb{Z}^+$ and $2\mathbb{N}$ are obviously countable since they are themselves subsets of $\mathbb{N}$; and $\mathbb{Z}$ is countable because we have just seen a bijection between it and $\mathbb{N}$.

What about the set of all rational numbers? Recall that $\mathbb{Q} = \{\frac{x}{y} \mid x, y \in \mathbb{Z},\ y \neq 0\}$. Surely there are more rational numbers than natural numbers? After all, there are infinitely many rational numbers between any two natural numbers. Surprisingly, the two sets have the same cardinality! To see this, let us introduce a slightly different way of comparing the cardinality of two sets.

If there is a one-to-one function $f : A \to B$, then the cardinality of $A$ is less than or equal to that of $B$. Now to show that the cardinality of $A$ and $B$ are the same we can show that $|A| \leq |B|$ and $|B| \leq |A|$. This corresponds

to showing that there is a one-to-one function $f : A \to B$ and a one-to-one function $g : B \to A$. The existence of these two one-to-one functions implies that there is a bijection $h : A \to B$, thus showing that $A$ and $B$ have the same cardinality. The proof of this fact, which is called the Cantor-Schröder-Bernstein theorem, is actually interesting, and we will skip it here — instead walking you through it on the homework. For us, this fact will be very useful: for example, to show that a set $S$ is countable, it is enough to give separate injections $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$, rather than designing a bijection (which is often trickier).

Back to comparing the natural numbers and the integers. First it is obvious that $|\mathbb{N}| \leq |\mathbb{Q}|$ because $\mathbb{N} \subseteq \mathbb{Q}$. So our goal now is to prove that also $|\mathbb{Q}| \leq |\mathbb{N}|$. To do this, we must exhibit an injection $f : \mathbb{Q} \to \mathbb{N}$. The following picture of a spiral conveys the idea of this injection:




Each rational number $\frac{a}{b}$ (written in its lowest terms, so that $\gcd(a,b) = 1$) is represented by the point $(a,b)$ in the infinite two-dimensional grid shown (which corresponds to $\mathbb{Z} \times \mathbb{Z}$, the set of all pairs of integers). Note that not all points on the grid are valid representations of rationals: e.g., all points on the $x$-axis have $b = 0$ so none are valid (except for $(0,0)$, which we take to represent the rational number 0); and points such as $(2,8)$ and $(-1,-4)$ are not valid either as the rational number $\frac{1}{4}$ is represented by $(1,4)$. But $\mathbb{Z} \times \mathbb{Z}$ certainly contains all rationals under this representation, so if we come up with an injection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N}$ then this will also be an injection from $\mathbb{Q}$ to $\mathbb{N}$ (why?).

The idea is to map each pair $(a,b)$ to its position along the spiral, starting at the origin. (Thus, e.g.,$(0,0) \to 0$, $(1,0) \to 1$, $(1,1) \to 2$, $(0,1) \to 3$, and so on.) It should be clear that this mapping maps every pair of integers injectively to a natural number, because each pair occupies a unique position along the spiral.

This tells us that $|\mathbb{Q}| \leq |\mathbb{N}|$. Since also $|\mathbb{N}| \leq |\mathbb{Q}|$, as we observed earlier, by the Cantor-Schröder-Bernstein Theorem $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.

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*Exercise.* Show that the set $\mathbb{N} \times \mathbb{N}$ of all ordered pairs of natural numbers is countable.

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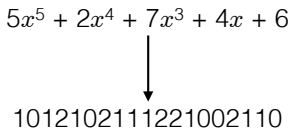
Our next example concerns the set of all binary strings (of any finite length), denoted $\{0,1\}^*$. Despite the fact that this set contains strings of unbounded length, it turns out to have the same cardinality as $\mathbb{N}$. To see this, we set up a direct bijection $f : \mathbb{N} \to \{0,1\}^*$ as follows. Note that it suffices to *enumerate* the elements of $\{0,1\}^*$ in such a way that each string appears exactly once in the list. We then get our bijection by setting $f(n)$ to be the $n$th string in the list. How do we enumerate the strings in $\{0,1\}^*$? Well, it's natural to list them in increasing order of length, and then (say) in *lexicographic* order (or, equivalently, numerically increasing order when viewed as binary numbers) within the strings of each length. This means that the list would look like

$$\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 1000, \ldots,$$

where $\varepsilon$ denotes the empty string (the only string of length 0). It should be clear that this list contains each binary string once and only once, so we get a bijection with $\mathbb{N}$ as desired.

Our final countable example is the set of all polynomials with natural number coefficients, which we denote $\mathbb{N}(x)$. To see that this set is countable, we give injections from $\mathbb{N}$ to $\mathbb{N}(x)$ and from $\mathbb{N}(x)$ to $\mathbb{N}$. The first of these is easy, since each natural number $n$ is itself already trivially a polynomial. For the injection the other way, from $\mathbb{N}(x)$ to $\mathbb{N}$, we will make use of (a variant of) the previous example. Note first that, by essentially the same argument as for $\{0,1\}^*$, we can see that the set of all *ternary* strings $\{0,1,2\}^*$ (that is, strings over the alphabet $\{0,1,2\}$) is countable. It therefore suffices to exhibit an injection $f : \mathbb{N}(x) \to \{0,1,2\}^*$, which in turn will give an injection from $\mathbb{N}(x)$ to $\mathbb{N}$.

How do we define $f$? Let's first consider an example, namely the polynomial $p(x) = 5x^5 + 2x^4 + 7x^3 + 4x + 6$. We can list the coefficients of $p(x)$ as follows: $(5,2,7,0,4,6)$. We can then write these coefficients as binary strings: $(101, 10, 111, 0, 100, 110)$. Now, we can construct a ternary string where a "2" is inserted as a separator between each binary coefficient (ignoring coefficients that are 0). Thus we map $p(x)$ to a ternary string as illustrated below:




It is easy to check that this is an injection, since the original polynomial can be uniquely recovered from this ternary string by simply reading off the coefficients between each successive pair of 2's. (Notice that this mapping $f : \mathbb{N}(x) \to \{0,1,2\}^*$ is not onto (and hence not a bijection) since many ternary strings will not be the image of any polynomials; this will be the case, for example, for any ternary strings that contain binary subsequences with leading zeros.)

Hence we have an injection from $\mathbb{N}(x)$ to $\mathbb{N}$, and from $\mathbb{N}$ to $\mathbb{N}(x)$, so $\mathbb{N}(x)$ is countable.

## 1.2 Cantor's Diagonalization

We have established that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ all have the same cardinality. What about $\mathbb{R}$, the set of real numbers? Surely they are countable too? After all, the rational numbers, like the real numbers, are dense (i.e., between any two rational numbers $a, b$ there is a rational number, namely $\frac{a+b}{2}$). In fact, between any two *real* numbers there is always a rational number. It is really surprising, then, that there are strictly more real numbers than rationals! That is, there is *no* bijection between the rationals (or the natural numbers) and the reals. We shall now prove this, using a beautiful argument due to Cantor that is known as *diagonalization*. In fact, we will show something even stronger: the real numbers in the interval $[0,1]$ are uncountable!

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*Exercise.* Show how to find a rational number between any two (distinct) real numbers.

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In preparation for the proof, recall that any real number can be written out uniquely as an infinite decimal with no trailing zeros. In particular, a real number in the interval $[0,1]$ can be written as $0.d_1d_2d_3\ldots$. In this representation, we write for example[1] 1 as $0.999\ldots$, and 0.5 as $0.4999\ldots$. (Thus rational numbers will always be represented as recurring decimals, while irrational ones will be represented as non-recurring ones. Importantly for us, all of these expressions will be infinitely long and unique.)

[1] To see this, write $x = .999\ldots$. Then $10x = 9.999\ldots$, so $9x = 9$, and thus $x = 1$.

**Theorem:** The real interval $\mathbb{R}[0,1]$ (and hence also the set of real numbers $\mathbb{R}$) is uncountable.

**Proof:** Suppose towards a contradiction that there is a bijection $f : \mathbb{N} \to \mathbb{R}[0,1]$. Then, we can enumerate the real numbers in an infinite list $f(0), f(1), f(2), \ldots$ as follows:

$$
\begin{array}{l}
f(0) = 0 .\textcircled{5}2\ 1\ 4\ 9\ 3\ 5\ 6\ \ldots \\
f(1) = 0 .\ 1\textcircled{4}1\ 6\ 2\ 9\ 8\ 5\ \ldots \\
f(2) = 0 .\ 9\ 4\textcircled{7}8\ 2\ 7\ 1\ 2\ \ldots \\
f(3) = 0 .\ 5\ 3\ 0\textcircled{9}8\ 1\ 7\ 5\ \ldots \\
\quad\vdots \qquad\qquad\qquad\quad \vdots
\end{array}
$$

Note that we have circled the digits on the *diagonal* of this list. This sequence of circled digits can be viewed as a real number, $r = 0.5479\ldots$, since it is an infinite decimal expansion.

Now consider the real number $s$ obtained by modifying every digit of $r$, say by replacing each digit $d$ with $d+1 \pmod{10}$ if $d \neq 9$, and with[2] 1 if $d = 9$; thus in our example above, $s = 0.6581\ldots$. We claim that $s$ does not occur in our infinite list of real numbers. Suppose that it did, and that it was the $n^{th}$ number in the list, $f(n)$. But by construction $s$ differs from $f(n)$ in the $(n+1)$th digit, so these two numbers cannot be equal! So we have constructed a real number $s$ that is not in the range of $f$. But this contradicts our original assertion that $f$ is a bijection. Hence the real numbers are not countable.

It is worth asking what happens if we apply the same method to $\mathbb{Q}$, in a (presumably futile) attempt to show that the rationals are uncountable. Well, suppose for contradiction that our bijective function $f : \mathbb{N} \to \mathbb{Q}[0,1]$ produces the following mapping:

$$
\begin{array}{l}
f(0) = 0 .\textcircled{1}4\ 0\ 0\ 0\ \ldots \\
f(1) = 0 .\ 5\textcircled{9}2\ 4\ 5\ \ldots \\
f(2) = 0 .\ 2\ 1\textcircled{4}2\ 1\ \ldots \\
\quad\vdots \qquad\qquad\quad \vdots
\end{array}
$$

Again, we consider the number $q$ obtained by modifying every digit of the diagonal as before, giving $q = 0.215\ldots$ in the above example. Again, by construction, the number $q$ does not appear in the list. However, this tells us nothing because we do not know that $q$ is rational (indeed, it is extremely unlikely for the decimal expansion to be periodic, as required for $q$ to be rational); hence the fact that $q$ is not in the list of rationals is *not* a contradiction! When dealing with the reals, the modified diagonal number was guaranteed to be a real number.

# 2 Self-Reference and Computability

Cantor's diagonalization argument turns out to be applicable far more generally than just for establishing the uncountability of the real numbers. (This once again strengthens the general EECS perspective towards mathematical thinking — the underlying concepts and proofs are almost always more important to us than the theorems.) However, one additional critical ingredient turns out to be required: the notion of "self-reference" (having a statement or program somehow be about itself). This has far-reaching consequences for the limits of computation (the Halting Problem) and the foundations of logic in mathematics (Gödel's

[2]The reason we treat $d = 9$ differently is that the same real number can have two decimal expansions; e.g., 0.999... = 1.000.

incompleteness theorem). Extremely recently, this has also turned out to have important consequences in physics due to the power of certain models of quantum computation.

## 2.1 The Liar's Paradox

Recall that propositions are statements that are either true or false. We saw in an earlier lecture that some statements are not well defined or too imprecise to be called propositions. But here is a statement that is problematic for more subtle reasons:

"All Cretans are liars."

So said a Cretan in antiquity, thus giving rise to the so-called liar's paradox which has amused and confounded people over the centuries. Why? Because if the statement above is true, then the Cretan was lying, which implies the statement is false. But actually the above statement isn't really a paradox; it simply yields a contradiction if we assume it is true, but if it is false then there is no problem.

A true formulation of this paradox is the following statement:

"This statement is false."

Is the statement above true? If the statement is true, then what it asserts must be true; namely that it is false. But if it is false, then it must be true. So it really is a paradox, and we see that it arises because of the self-referential nature of the statement. Around a century ago, this paradox found itself at the center of foundational questions about mathematics and computation.

We will now study how this paradox relates to computation. Before doing so, let us consider another manifestation of the paradox, attributed to the great logician Bertrand Russell (but actually a version of a related paradox devised by Russell). In a village with just one barber, every man keeps himself clean-shaven. Some of the men shave themselves, while others go to the barber. The barber proclaims:

"I shave all and only those men who do not shave themselves."

It seems reasonable then to ask the question: Does the barber shave himself? Thinking more carefully about the question though, we see that, assuming that the barber's statement is true, we are presented with the same self-referential paradox: a logically impossible scenario. If the barber does not shave himself, then according to what he announced, he shaves himself. If the barber does shave himself, then according to his statement he does not shave himself!

(Of course, the real resolution to the barber paradox is simple: the barber is a woman who doesn't need to shave. But that's not the point.)

## 2.2 The Halting Problem

Are there tasks that a computer cannot perform? For example, we would like to ask the following basic question when compiling a program: does it run forever, i.e. go into what feels like an infinite loop? In 1936, Alan Turing showed that there is no program that can perform this test. The proof of this remarkable fact is very elegant and combines two ingredients: self-reference (as in the liar's paradox), and the fact that we cannot separate programs from data. In computers, a program is represented by a finite string of bits just as integers, characters, and other data are. The only difference is in how the string of bits is interpreted.

We will now examine the Halting Problem. Given the description of a program and its input, we would like to know if the program ever halts when it is executed on the given input. In other words, we would like to write a program `TestHalt` that behaves as follows:

$$\texttt{TestHalt(P,x)} = \begin{cases} \text{“yes”}, & \text{if program } P \text{ halts on input } x \\ \text{“no”}, & \text{if program } P \text{ loops on input } x \end{cases}$$

Why can't such a program exist? First, let us use the fact that a program is just a bit string, so it can be input as data. This means that it is perfectly valid to consider the behavior of `TestHalt(P,P)`, which will output "yes" if $P$ halts on $P$, and "no" if $P$ loops forever on $P$. We now prove that such a program cannot exist.

**Theorem:** *The Halting Problem is uncomputable; i.e., there does not exist a computer program TestHalt with the behavior specified above on all inputs* $(P,x)$. *(Note that this statement holds regardless of what hardware or programming language we use.)*

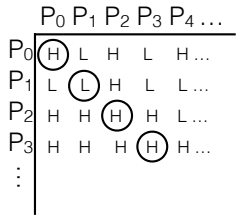
**Proof:** Assume for contradiction that the program `TestHalt` exists. Then we can easily use it as a subroutine to construct the following program:

```
Turing(P)

  if TestHalt(P,P) = "yes" then loop forever

  else halt
```

So if the program $P$ when given $P$ as input halts, then `Turing(P)` loops forever; otherwise, `Turing(P)` halts. Note that the program `Turing` is very easy to construct if we are given the program `TestHalt`.

Now let us look at the behavior of `Turing(Turing)`. There are two cases: either it halts, or it does not. If `Turing(Turing)` halts, then it must be the case that `TestHalt(Turing,Turing)` returned "no." But by definition of `TestHalt`, that would mean that `Turing(Turing)` should not have halted. In the second case, if `Turing(Turing)` does not halt, then it must be the case that `TestHalt(Turing, Turing)` returned "yes," which would mean that `Turing(Turing)` should have halted. In both cases, we arrive at a contradiction which must mean that our initial assumption, namely that the program `TestHalt` exists, was wrong. Thus, `TestHalt` cannot exist, so it is impossible for a program to definitively check if any general program halts! □

What proof technique did we use? This was actually a proof by diagonalization, the same technique that we used earlier to show that the real numbers are uncountable! Why? Since the set of all computer programs is countable (they are, after all, just finite-length strings over some alphabet, and the set of all finite-length strings is countable), we can enumerate all programs as follows (where $P_i$ represents the $i^{\text{th}}$ program):



The $(i,j)^{\text{th}}$ entry in the table above is H if program $P_i$ halts on input $P_j$, and L (for "Loops") if it does not halt. (Here, we assume that malformed programs with syntax errors just halt with an error.) Now if the program `Turing` exists it must occur somewhere on our list of programs, say as $P_n$. But this cannot be, since if the $n^{\text{th}}$ entry in the diagonal is H, meaning that $P_n$ halts on $P_n$, then by its definition `Turing` loops on $P_n$; and if the entry is L, then by definition `Turing` halts on $P_n$. Thus the behavior of `Turing` is different from that of $P_n$, and hence `Turing` does not appear on our list. Since the list contains all possible programs, we must conclude that the program `Turing` does not exist. And since `Turing` is constructed by a simple modification of `TestHalt`, we can conclude that `TestHalt` does not exist either. Hence the Halting Problem cannot be solved.

In fact, there are many more questions we would like to answer about programs but cannot answer decisively. For example, we cannot definitively know if a program ever outputs anything or if it ever executes a specific line. We also cannot definitively check if two programs produce the same output. And we cannot definitively check to see if a given program is a virus. To attempt to do any of these tasks, we have to allow our approach to answer "I don't know" and give up. Otherwise, the halting problem's impossiblity would infect our approach as well. These issues are explored in greater detail in the advanced course CS172 (Computability and Complexity).

## 2.3 Uncomputable numbers

The fact that the real numbers are uncountably infinite and that there are only a countable number of computer programs tells us that the vast majority of real numbers are fundamentally unknowable to computers. The halting problem above tells us that many of them are also interesting. For example, consider the real number between 0 and 1 whose *i*th binary digit is 0 if the *i*th computer program doesn't halt and is 1 if the *i*th computer program does halt. The halting problem argument tells us that this number is uncomputable.

In a very related proof, the logician Gödel showed that the following real number is also uncomputable. Consider all finite strings of mathematical symbols involving $\forall, \exists$, variables, as well as the arithmetic operations $+, *, -, /$ and exponentiation, comparisons $=, <, >$ and the logical operators $\neg, \wedge, \vee, \implies$. A string like that is either a syntax error or it is a valid proposition about the natural numbers. All such finite strings are certainly countable. So we can talk about the *i*th such string. Consider the real number between 0 and 1 in which the *i*th digit is 0 if the string is a syntax error or the proposition it represents is false. The *i*th digit is 1 if the string is well-formed and the proposition it represents is true (i.e. there is no counterexample to it). The resulting real number is uncomputable.

This turns out to mean that there are true statements about the integers for which there is no proof. In a very real sense, they just happen to be true for no good[3] reason. The even more surprising consequence is that the same uncomputability result holds if we replace "true" and "false" with "provable" and "unprovable." Here, we can consider a proposition provable if it is either a syntax error (syntax can be checked by a program), has a counterexample, or has a valid proof. It is unprovable otherwise. So, not only are there lots of unprovable assertions out there (if there were a finite number, we could simply have a finite list of them that a computer program could check) but they are impossible for a computer program to reliably recognize[4] as unprovable.

The proof of these facts is beyond the scope of the course, but is related to the deep connection between

[3] To be more precise, they are true because they are true in the specific concrete model you have for the integers.

[4] Being a valid proof can be checked by a computer program. Each statement has to follow logically from those that came before. The number of potential proofs is countably infinite. The problem is that the program that simply compares statements to all valid proofs is not guaranteed to halt! Because there are unprovable statements, it might just run forever and never encounter neither a proof nor a counterexample.

proofs and computation. Computation is a kind of living proof.