"Theorem": All students love CS70.

"Proof:" Let $P(n)$ be "given a set of n students, they all love CS 70".

<u>Base case</u>: $P(0)$ is trivially true.

<u>Inductive Step</u>:

Assume $P(n)$ is true.

Suppose we're given a set of students $\{S_1, S_2, \dots, S_n, S_{n+1}\}$.

By inductive hypothesis, students in $\{S_1, \dots, S_n\}$ all love CS70. Similarly, students in $\{S_2, \dots, S_{n+1}\}$ all love CS70.

$\Rightarrow$ $S_1, \dots, S_{n+1}$ all love CS70.

By the principle of induction, since $P(0)$ is true and $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$, we know the statement is true. $\square$

Q: Do you agree? What's wrong?

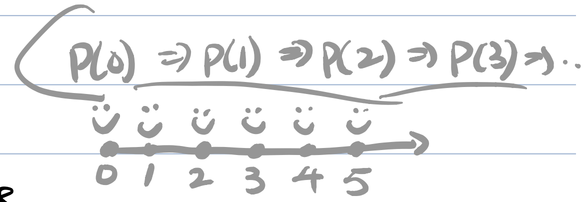
$P(0) \not\Rightarrow P(1)$

so we didn't really prove "$\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$".

1. Induction

Induction is a technique for proving $\forall n \in \mathbb{N},\ P(n)$.

1.1 Simple Induction



To prove $P(n)$ is true for all $n \in \mathbb{N}$, use

<u>Base case</u>: check $P(0)$ holds

<u>Inductive Step</u>: Show $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$.

<u>E.g.</u> Prove that $\sum_{j=0}^{n} ar^j = \dfrac{ar^{n+1}-a}{r-1}$ where $\cancel{a}$ $a, r \in \mathbb{R}, r \neq 1, n \in \mathbb{N}$.

<u>Pf</u>: Let $P(n)$ be the statement $\sum_{j=0}^{n} ar^j = \dfrac{ar^{n+1}-a}{r-1}$.

<u>Base case</u>: $P(0)$ holds, because $\sum_{j=0}^{0} ar^j = a = \dfrac{ar-a}{r-1}$ $a(r-1)$

<u>Inductive Step</u>: WTS $\forall k \in \mathbb{N}$, $P(k) \Rightarrow P(k+1)$.

Inductive hypothesis: Assume $P(k)$ is true, i.e.

assume $\sum_{j=0}^{k} ar^j = \dfrac{ar^{k+1}-a}{r-1}$

want to show $\sum_{j=0}^{k+1} ar^j = \dfrac{ar^{k+2}-a}{r-1}$.

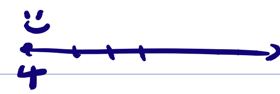
$$\sum_{j=0}^{k+1} ar^j = \sum_{j=0}^{k} ar^j + ar^{k+1}$$

$$\overset{IH}{=} \frac{ar^{k+1}-a}{r-1} + \frac{ar^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} \cancel{- ar^{k+1}}}{r-1}$$

$$= \frac{ar^{k+2}-a}{r-1}$$

$\square$



E.g. Prove that $2^n < n!$ for every integer $n \geq 4$.

Pf: Let $P(n)$ be the statement $2^n < n!$

Base case: $P(4)$ holds, because $2^4 = 16 < 4! = 24$

Inductive Step: WTS $\forall k \in \mathbb{N},\ k \geq 4,\ P(k) \Rightarrow P(k+1)$.

Assume $P(k)$, i.e. assume $2^k < k!$ for some integer $k \geq 4$.

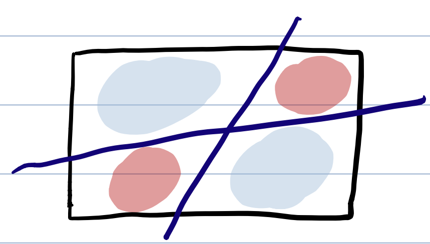
Want to show $2^{k+1} < (k+1)! = k!\,(k+1)$

Notice that

$$\begin{aligned} 2^{k+1} &= 2^k \times 2 \\ &\overset{IH}{<} k! \times 2 \\ &< k! \times (k+1) \\ &= (k+1)! \end{aligned}$$

$\square$

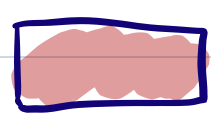
E.g.



Prove a map with n lines is 2-colorable, where $n \in \mathbb{N}$.

Pf: Let $P(n)$ be the statement a map with n lines is 2-colorable.

Base case: $P(0)$ holds, because we can color the entire map using a single color.



Inductive Step: WTS $\forall k \in \mathbb{N},\ P(k) \Rightarrow P(k+1)$.

Assume $P(k)$, i.e. assume a map with k lines is

2-colorable.

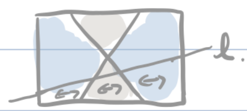
added after lecture for completeness

Want to show a map with $k+1$ lines is 2-colorable.

Given a map with $k+1$ lines, remove a line ℓ.

By IH, the new map is 2-colorable.

Color the new map.



Add ℓ back.

ℓ divides the colored map into two sides.

Pick one side and swap the colors.

The result is still a valid coloring, because for each shared border, it's either ℓ or not ℓ.

① if it's not ℓ, two sides have different colors by IH;

② if it's ℓ, two sides now have different colors because of the swap. □

E.g. Prove the sum of the first k odd numbers is a perfect square.

Pf: Let $P(n)$ be the statement $\sum_{x=1}^{n} (2x-1) = m^2$ for some $m \in \mathbb{Z}^+$

Inductive Step:

Assume $P(k)$ holds for some $k \in \mathbb{Z}^+$, i.e.

$$\sum_{x=1}^{k} (2x-1) = m^2 \text{ for some } m, k \in \mathbb{Z}^+$$

Want to show $P(k+1)$ holds, i.e. $\sum_{x=1}^{k+1} (2x-1)$

$$= \sum_{x=1}^{k} (2x-1) + (2k+1) = m^2 + 2k + 1.$$

is a perfect square ???

$[1=1,\ 1+3=4=2^2,\ 1+3+5=9=3^2,\ \dots]$

Let $P(n)$ be the statement $\sum_{x=1}^{n}(2x-1)=n^2$.

__Base case__: $P(1)$ holds because $1=1=1^2$.

__Inductive Step__: Assume $\sum_{x=1}^{k}(2x-1)=k^2$ for some $k\in\mathbb{Z}^+$.

Then
$$\sum_{x=1}^{k+1}(2x-1)=\left(\sum_{x=1}^{k}(2x-1)\right)+2k+1 \overset{IH}{=} k^2+2k+1 = (k+1)^2. \qquad \square$$

1.2 Strong Induction

To prove $P(n)$ is true for all $n\in\mathbb{N}$, use

__Base case__: check $P(0)$ holds.

__Inductive Step__: Show $\forall k\in\mathbb{N},\ [P(0)\wedge\cdots\wedge P(k)]\Rightarrow P(k+1)$.

__E.g.__ Prove that if n is an integer greater than 1, then n can be written as a product of primes.

__Pf__: __Base case__: 2 is a prime and a product of itself, so the statement holds for $n=2$.

__Inductive Step__: Assume all integers $2\le j\le k$ can be written as a product of primes.

Consider $k+1$. If $k+1$ is prime, we're done.

Otherwise, $k+1=ab$ for some integers a,b with $2\le a,b<k+1$.

By IH, a, b can be written as a product of primes.

Hence, $k+1$ can be written as a product of primes. $\square$

E.g. Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

$\forall n \geq 12, P(n)$.

Pf: Let $P(n)$ be the statement $n = 4x + 5y$ for some $x, y \in \mathbb{N}$.

Base case: $P(12)$ is true $12 = 4 \times 3 + 5 \cdot 0$.

$P(13), P(14), P(15)$ holds because ...

Inductive Step:

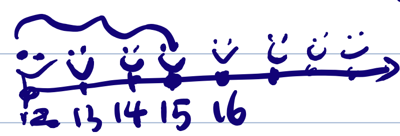
Assume $P(n)$ holds for all $12 \leq n \leq k$ for some $k \geq 15$.

Consider $k+1$

[If $\boxed{(k+1)-4} = 4x+5y$, $k+1 = 4(x+1) + 5y$.

but need $(k+1)-4 \geq 12$, i.e $k \geq 15$]

Since $(k+1) - 4 \geq 12$, by IH, $(k+1) - 4 = 4x + 5y$ for $x, y \in \mathbb{N}$, so $k+1 = 4(x+1) + 5y$.



Rem. **Well-ordering Principle** states $S \subseteq \mathbb{N}$, $S \neq \emptyset$, then S has a least element.

The validity of the principle of induction and strong induction follows from WOP.

1.3 Recursion

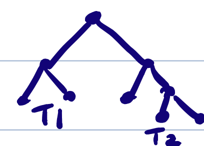
To prove a statement holds for recursively defined objects, use

<u>Base case</u>: the result holds for all elements specified in the base case

<u>Recursive Step</u>: Show if the statement holds for each element used to construct new elements, then it holds for these new elements.

<u>E.g.</u> Binary trees can be constructed recursively.

Define height $h(T)$ recursively



<u>Base case</u> (T = root) : $h(T) = 0$

<u>Recursive Step</u> ($T = T_1 \cdot T_2$) : $h(T) = 1 + \max(h(T_1), h(T_2))$

Define number of vertices $n(T)$ recursively

<u>Base case</u> (T = root) : $n(T) = 1$.

<u>Recursive Step</u> ($T = T_1 \cdot T_2$) : $n(T) = 1 + n(T_1) + n(T_2)$.

Prove $n(T) \leq 2^{h(T)+1} - 1$ for any binary tree T.

<u>Pf</u>: <u>Base case</u>: T = root. Then $n(T) = 1$ and $h(T) = 0$.

Statement holds because $1 \leq 2^{0+1} - 1 = 1$

<u>Recursive Step</u>: Consider $T = T_1 \cdot T_2$.

want to show $n(T) \leq 2^{h(T)+1} - 1$.

Notice that

$$
\begin{aligned}
n(T) &\overset{\text{def}}{=} 1 + n(T_1) + n(T_2) \\
&\overset{\text{IH}}{\leq} 1 + \left(2^{h(T_1)+1} - 1\right) + \left(2^{h(T_2)+1} - 1\right) \\
&= 2^{h(T_1)+1} + 2^{h(T_2)+1} - 1 \\
&\leq 2 \cdot \max\left(2^{h(T_1)+1}, 2^{h(T_2)+1}\right) - 1 \\
&= 2 \cdot 2^{\max(h(T_1), h(T_2))+1} - 1 \\
&\overset{\text{def}}{=} 2 \cdot 2^{h(T)} - 1 \\
&= 2^{h(T)+1} - 1.
\end{aligned}
$$

$\square$