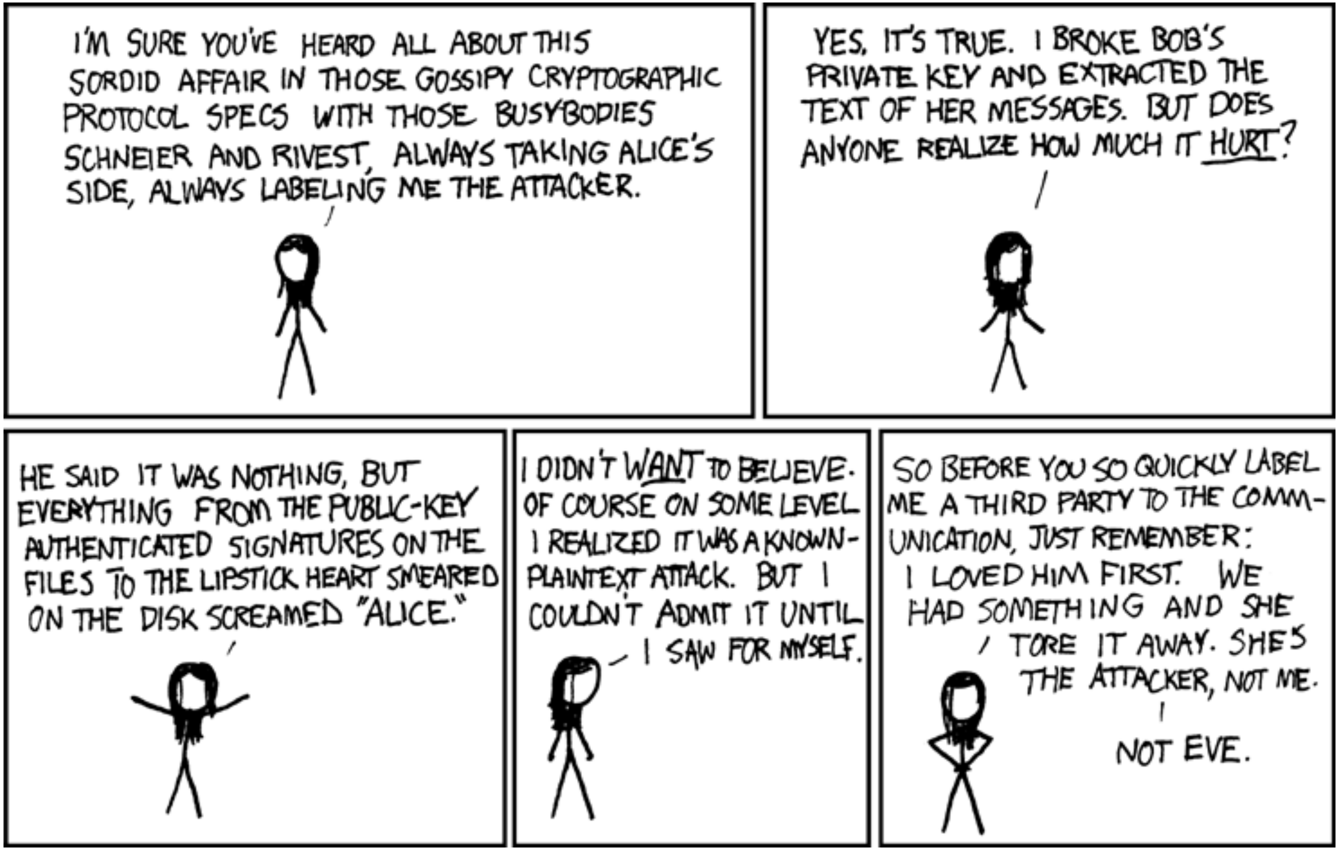
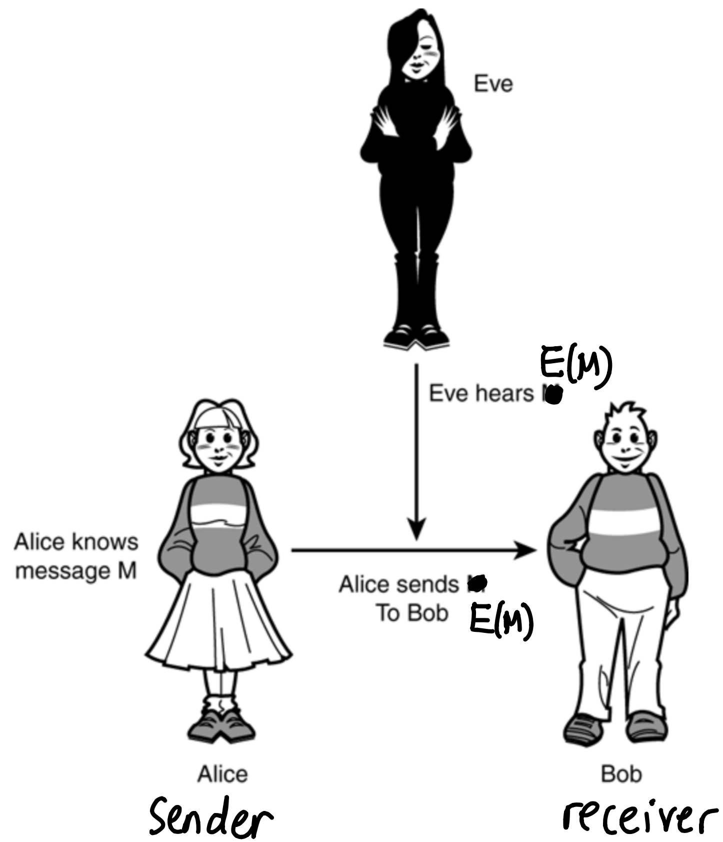
Lecture 10: Cryptography



Credit: https://xkcd.com/177/

Credit: Sagnik!

Basic Setup



Credit: `https://flylib.com/books/en/1.581.1.188/1/`

Recall: XOR

Recall the XOR operation:

x	$\cancel{y}$ b	x $\oplus$ $\cancel{y}$ b	$(x \oplus b) \oplus b$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

Notice that for any bits x, b we have $(x \oplus b) \oplus b = x$

One-Time Pad

Alice (the sender) wants to send a n-bit message m to Bob (the receiver).

Setup:

- Alice and Bob generate a random key k.

Encryption:

Decryption:

Notice that $D(E(m)) = (m \oplus k) \oplus k = m$, i.e. Bob always receives the message Alice sent.

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- And every single user would've had to do this.

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- **public keys**: everyone knows!
- **private keys**: only Bob knows.

RSA Protocol

Everyone can send messages to Bob.
For now, let's say Alice wants to send a message m to Bob.

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- Bob chooses two large (2048-bit) distinct primes p, q.

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```
>>> c^d % N
>>>
```

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We need to analyze:

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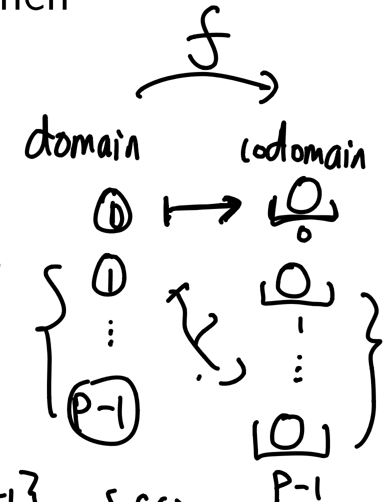
- Correctness: $D(E(m)) = m$?
- Efficiency: Can Alice and Bob perform their steps efficiently?
- Security: Can Eve break it?

Fermat's Little Theorem

Theorem: Let p be a prime and $a \not\equiv 0 \pmod{p}$. Then

Goal: $\underbrace{a^{p-1}}$ $\equiv 1 \pmod{p}$.



Proof. $f : \{0, 1, 2, \ldots, p-1\} \to \{0, 1, \ldots, p-1\}$

$x \mapsto ax \bmod p$

is a bijection.

Since $f(0) = 0 \cdot a \bmod p = 0 \bmod p = 0$, $\underbrace{\{1, 2, \ldots, p-1\}} = \underbrace{\{f(1), \ldots, f(p-1)\}}$.

$\underbrace{\forall x = 1, \ldots, p-1}$, $\underset{\substack{\| \\ ax \bmod p}}{f(x)} \equiv ax \pmod{p}$.

$$\Rightarrow \underline{\prod_{x=1}^{p-1} x} = \prod_{x=1}^{p-1} f(x) \equiv \prod_{x=1}^{p-1} (ax) = \underline{a^{p-1} \prod_{x=1}^{p-1} x} \pmod{p}$$

Since p is a prime, $\forall x = 1, \ldots, p-1$, $\gcd(x, p) = 1$. $\Rightarrow x^{-1}$ modulo p exists.

$$\left(\prod_{x=1}^{p-1} x^{-1}\right)\cancel{\left(\prod_{x=1}^{p-1} x\right)} \equiv a^{p-1} \cancel{\left(\prod_{x=1}^{p-1} x\right)\left(\prod_{x=1}^{p-1} x^{-1}\right)} \pmod{p} \Rightarrow 1 \equiv a^{p-1} \pmod{p}$$

Goal: $D(E(m)) = m$.

$$(m^e \% N)^d \% N \not\equiv m$$

Notice that $0 \le D(E(m)) \le N-1$,

so only need to show $D(E(m)) \equiv m \pmod N$.

m, n are coprime.

$$X \equiv 3 \pmod n$$

$$X \equiv 3 \pmod m$$

Find me a solution!!!

$x = 3$

Find me all solutions!!

$3 + (mn)k, \quad k \in \mathbb{Z}$

$$E(m) = m^e \% N \equiv m^e \pmod N$$

$$D(c) = c^d \% N \equiv c^d \pmod N$$

$$D(\underbrace{E(m)}_{c}) \equiv E(m)^d \equiv (m^e)^d = m^{ed} \pmod N$$

$$\boxed{\text{Goal: } m^{ed} \equiv m \pmod N}$$

RSA correctness

FLT: prime p, and $m \not\equiv 0 \pmod p$, $m^{p-1} \equiv 1 \pmod p$

Theorem: Let D, E be the RSA decryption and RSA encryption functions respectively. Then $D(E(m)) = m$, i.e. RSA protocol always decrypts correctly.

$m^{ed} \equiv m \pmod N$. $N = pq$

Proof. Let $x = m^{ed}$. $\boxed{\text{Goal}: x \equiv m \pmod N}$.

Since $ed \equiv 1 \pmod{(p-1)(q-1)}$, so $\exists k \in \mathbb{Z}$, $ed - 1 = k(p-1)(q-1)$.

Then $x = m^{1+k(p-1)(q-1)}$

If $m \not\equiv 0 \pmod p$, $m^{p-1} \equiv 1 \pmod p$ $\Rightarrow x = m \cdot \overbrace{m^{k(p-1)(q-1)}}^{1} \equiv m \pmod p$.

If $m \equiv 0 \pmod p$, $x \equiv 0 \equiv m \pmod p$.

Thus, $\boxed{\begin{cases} x \equiv m \pmod p \\ x \equiv m \pmod q \end{cases}}$

Notice that $x = m$ is a solution.

Since p, q are primes, i.e. $\gcd(p, q) = 1$, by CRT, the solution is unique modulo $N = pq$. i.e. $x \equiv m \pmod N$.

RSA Efficiency

Setup

- Bob chooses two large distinct primes p and q.
  how???

e s.t. $\gcd(e, (p-1)(q-1)) = 1$

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$e^{-1} \bmod (p-1)(q-1).$

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- ...but how to check primes?
- there is an efficient algorithm that tests if N is prime (polynomial time in the number of bits of N).

RSA Security

Cryptograph relies on assumptions.

RSA Assumption: Given N, e, and $m^e \mod N$, there is no efficient algorithm for finding m.

We believe Eve cannot break RSA.

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- But prime factorization is hard!
- For large N, no efficient, non-quantum algorithm is known.

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- Eve sends $E(m)$ to Amazon.
- Now Eve can use my credit card.

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- If Amazon gets same message twice, reject.

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- Bob could prove his identity by showing Alice d, but he doesn't want to do that.
- Alice chooses a message m and asks Bob to send her m^d mod N.
- Alice can verify $(m^d)^e \equiv m \pmod{N}$.

$$E(D(m)) = D(E(m)) = m$$

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- Now Eve knows $(r^e \underbrace{E(m)}_{= m^e})^d \equiv \underbrace{r^{ed} m^{ed}}_{= (rm)^{ed}} \equiv rm \pmod{N}$.

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- Now Eve knows $(r^e E(m))^d \equiv r^{ed} m^{ed} \equiv rm \pmod{N}$.
- Eve knows r; so Eve computes $r^{-1} \mod N$ to recover m.

THE END!

Thank you for coming!