Continuous Probability

Lec. 21

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Continuous Probability

Recall that a random variable is a function $X : \Omega \to \mathbb{R}$.
When we write $X = k$, it actually means $\{\omega \in \Omega : X(\omega) = k\}$.

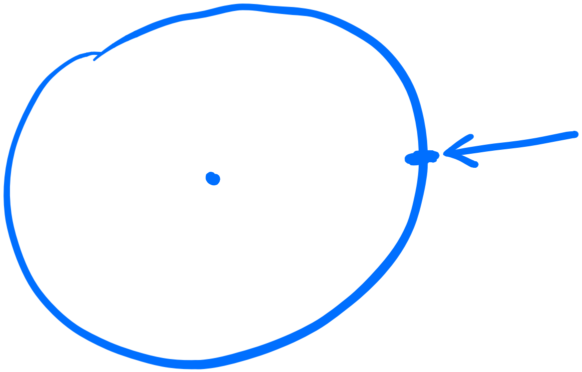
When you flip a coin, $|\Omega| = 2$. For a Poisson R.V., $|\Omega| = |\mathbb{N}|$.

H T

In the real world, we are often more interested in sample spaces that are uncountably infinite in size.

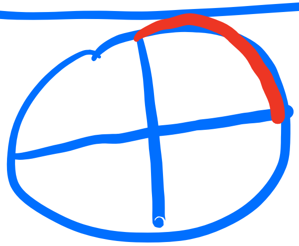
Continuous Probability

Circumference: 2



X is a continuous R.V. $\quad X \in [0,2)$

$$P(X=0) = 0$$



$$P(\quad) = 1/4$$

Continuous Probability

Consider a gameshow with a wheel with circumference 2. If the contestant spins the wheel, and it lands at the exact same point it started at, the contestant wins a million dollars.

We can model this as X is some random variable taking on values in $[0, 2) \subset \mathbb{R}$. The probability of the contestant winning then would correspond to $\mathbb{P}(X = 0)$.

If we assign $\mathbb{P}(X = 0)$ a positive probability, then there are *uncountably* infinitely many other events $X = k, k \in [0, 2)$ that have the same probability. Summing over all possible outcomes, the total probability would end up being greater than 1.

Thus, in the continuous case, $\boxed{\mathbb{P}(X = \omega) = 0 \text{ for all } \omega \in \Omega.}$

Probability Density Function (PDF)

A probability density function (pdf) for a real-valued random variable X is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

- $\forall x \in \mathbb{R}, \quad f(x) \geq 0$ (non negative)
- $\int_{-\infty}^{\infty} f(x)dx = 1$

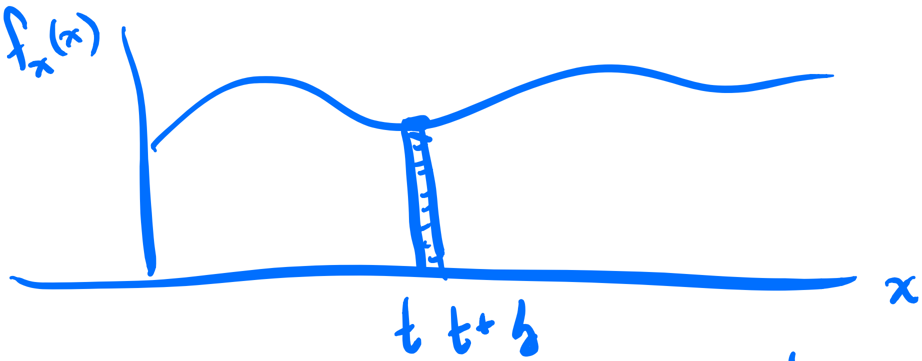
$\mathbb{P}(a < X < b)$

and:

$$\underbrace{\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx}$$

$$\underline{\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f(x)\,dx = 0} \checkmark$$

Probability Density?



if δ small

$$P(t \le X \le t+\delta) = \int_t^{t+\delta} f_X(x)\,dx \approx f(t)\cdot\delta$$

$$f(t) = \frac{P(t \le X \le t+\delta)}{\delta}$$

$\leftarrow$ probability

$\leftarrow$ length

"probability density"

Probability Density?

Consider a tiny interval $(t, t+\delta)$.

$$\mathbb{P}(t \leq X \leq t+\delta) = \int_t^{t+\delta} f(x)dx$$

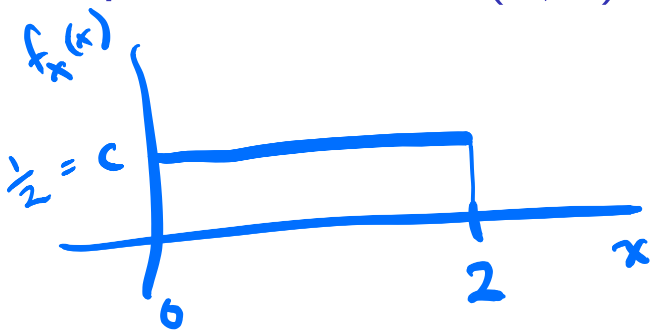
Since the interval is tiny, f is approximately constant on the interval:

$$\int_t^{t+\delta} f(x)dx \approx f(t) \cdot \delta$$

So,

$$\mathbb{P}(t \leq X \leq t+\delta) \approx f(t) \cdot \delta \to f(t) \approx \frac{\mathbb{P}(t \leq X \leq t+\delta)}{\delta}$$

Example: $X \sim \mathit{Unif}(0,2)$



$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ c & \text{if } x \in [0,2] \\ 0 & \text{if } x > 2 \end{cases}$$

What is the pdf of X?

We know by def. of valid pdf

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_0^2 c\,dx = 1$$

$$2c = 1 \implies c = \frac{1}{2}$$

Example: $X \sim Unif(0, 2)$

The pdf of X must be a constant c on $[0, 2]$, since all values in the interval are equally likely.

$$f(x) = \begin{cases} 0 & \forall x \in (-\infty, 0) \\ c & \forall x \in [0, 2] \\ 0 & \forall x \in (2, \infty) \end{cases}$$

We can find c by enforcing the constraint that the pdf integrates to 1.

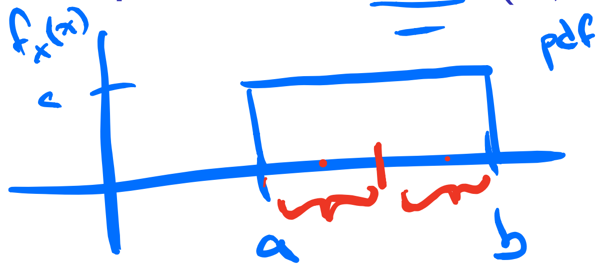
$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$c \leftarrow \int_0^2 cdx = 1$$

$$2c = 1$$

$$\rightarrow c = \frac{1}{2}$$

Example: $X \sim Unif(a, b)$



$$f(x) = \begin{cases} 0 & \forall x \in (-\infty, a) \\ \frac{1}{b-a} & \forall x \in [a, b] \\ 0 & \forall x \in (b, \infty) \end{cases}$$

what does uniformly random mean?

It means probability is proportional to length.

Analogs

The majority of definitions and techniques from the discrete case transfer over to the continuous case by simply swapping summations for integrals, and replacing the pmf with the pdf.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

pdf

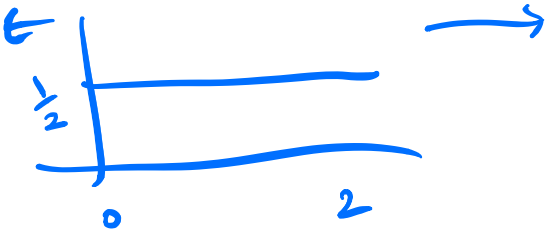
Discrete.

$$\mathbb{E}[X] = \sum_x x \, \mathbb{P}(X = x)$$

$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x) dx - \left(\int_{-\infty}^{\infty} x f(x) dx\right)^2
\end{aligned}$$

$\mathbb{E}[X^n]$ is called the n^{th} moment

Example: $X \sim Unif(0,2)$



What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f_X(x)\, dx$$

$$= \int_0^2 x \cdot \frac{1}{2}\, dx = \frac{x^2}{2} \cdot \frac{1}{2} \Big|_0^2 = \frac{4}{4} - 0 = 1$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \, f_X(x)\, dx = \int_0^2 x^2 \cdot \frac{1}{2}\, dx = \frac{8}{6}$$

$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{8}{6} - 1^2 = \frac{1}{3}$$

Example: $X \sim Unif(0,2)$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{2} x\frac{1}{2}dx = 1$$

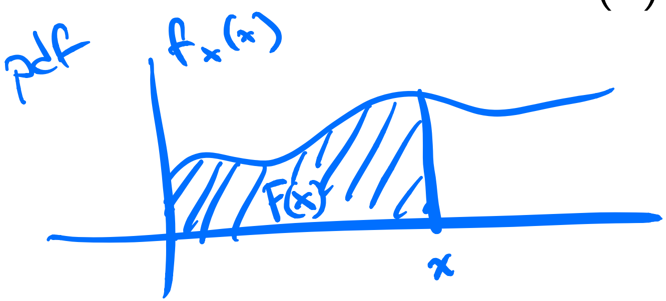
$$\begin{aligned}
\mathrm{Var}[X] &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \int_{-\infty}^{\infty} x^2 f(x)dx - (\int_{-\infty}^{\infty} xf(x)dx)^2 \\
&= \int_{0}^{2} x^2 \frac{1}{2}dx - 1^2 = \frac{1}{2}(\frac{x^3}{3}\Big|_0^2) - 1 \\
&= \frac{4}{3} - 1 = \frac{1}{3}
\end{aligned}$$

Cumulative Distribution Function (CDF)

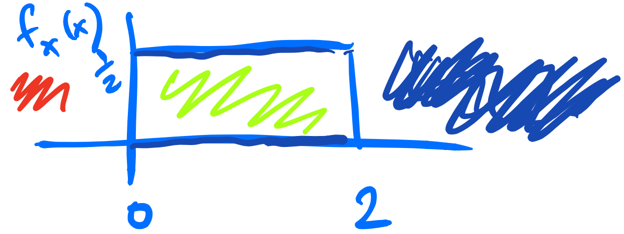
The CDF is a bridge between the discrete and continuous cases.

The cumulative distribution function (cdf) of a random variable X is the function F where:

$$F(x) = \mathbb{P}(X \leq x)$$



Example: $X \sim Unif(0,2)$



What is the CDF of X?

$$F(x) = \underbrace{\mathbb{P}(X \le x)}_{} = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_0^x \frac{1}{2}\,dx = \frac{1}{2}x \Big|_0^x = \frac{x}{2}$$

$\left(\text{if } x \in [0,2]\right)$

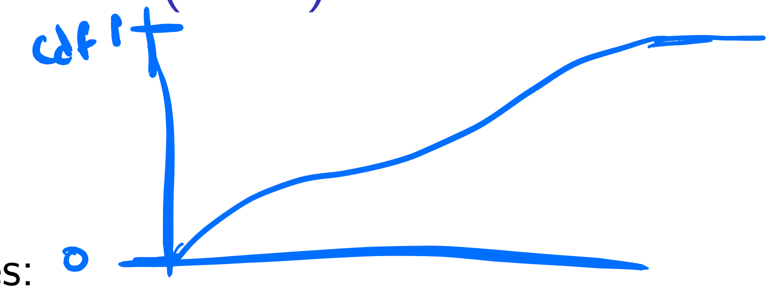
$$\mathbb{P}(X \le x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{2} & \text{if } x \in [0,2] \\ 1 & \text{if } x > 2 \end{cases}$$

Example: $X \sim Unif(0,2)$

$$F(x) = \mathbb{P}(X \leq x)$$

$$= \int_{-\infty}^{x} f(s)ds = \begin{cases} 0 & \forall x \in (-\infty, 0) \\ \frac{1}{2}s\Big|_0^x = \frac{1}{2}x - 0 = \underline{\frac{x}{2}} & \forall x \in [0,2] \\ 1 & \forall x \in (2, \infty) \end{cases}$$

Cumulative Distribution Function (CDF)



The cdf has a few key properties:

- $\lim_{x \to -\infty} F(x) = 0$ ✓
- $\lim_{x \to +\infty} F(x) = 1$ ✓
- It is *monotonically increasing*

Furthermore, the CDF uniquely characterizes the distribution of the random variable.

$$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$$ by FTC

Example: $X \sim Unif(0,2)$

Recall the CDF of X.

$$F(x) = P(X \le x) = \begin{cases} 0 & \forall x < 0 \\ \frac{x}{2} & \forall x \in [0,2] \\ 1 & \forall x > 2 \end{cases}$$

$$\lim_{x \to -\infty} F(x) = 0 \checkmark$$

$$\lim_{x \to +\infty} F(x) = 1 \checkmark$$

Monotonically increasing $\checkmark$

Example: $X \sim Unif(0,2)$

The cdf of X is:

$$F(x) = P(X \leq x) = \begin{cases} 0 & \forall x \in (-\infty, 0) \\ \frac{x}{2} & \forall x \in [0,2] \\ 1 & \forall x \in (2, \infty) \end{cases}$$

- $\lim_{x \to -\infty} F(x) = 0$ ✓
- $\lim_{x \to +\infty} F(x) = 1$ ✓
- It is *monotonically increasing* ✓

Recovering the PMF/PDF from the CDF

In the continuous case, take a derivative:

$$f(x) = \frac{dF(x)}{dx}$$ (by FTC)

In the discrete case, "discrete derivative":

$$\mathbb{P}(x) = \frac{F(x) - F(x-1)}{x - (x-1)}$$

Example: $X \sim Unif(0,2)$

cdf to pdf?

$$P(X \le x) = \begin{cases} 0 & x < 0 \\ \frac{x}{2} & x \in [0,2] \\ 1 & x > 2 \end{cases}$$

derivative

integrate

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x \in [0,2] \\ 0 & x > 2 \end{cases}$$

Example: $X \sim Unif(0, 2)$

The cdf of X is:

$$P(X \leq x) = \begin{cases} 0 & \forall x \in (-\infty, 0) \\ \frac{x}{2} & \forall x \in [0, 2] \\ 1 & \forall x \in (2, \infty) \end{cases}$$

Taking derivative with respect to x for each part of the domain, the pdf of X is:

$$f(x) = \begin{cases} 0 & \forall x \in (-\infty, 0) \\ \frac{1}{2} & \forall x \in [0, 2] \\ 0 & \forall x \in (2, \infty) \end{cases}$$

Conditioning on an Event

X is a continuous r.v.

A is an event.

Let $\mathcal{A}$ be the set of values $X(\omega)$ for all $\omega \in A$.

$$\underbrace{f_{X|A} \cdot \cancel{\delta}} = \mathbb{P}(x \le X \le x+\delta \mid X \in \mathcal{A})$$

$$= \frac{\mathbb{P}(x \le X \le x+\delta \wedge X \in \mathcal{A})}{\mathbb{P}(X \in \mathcal{A})}$$

$$= \begin{cases} \dfrac{f_X(x) \cdot \cancel{\delta}}{\mathbb{P}(A)} & \forall\, x \in A \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \boxed{f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(A)} & \forall\, x \in \mathcal{A} \\ 0 & \text{otherwise.} \end{cases}}$$

Discrete Case.

X is discrete.

$$\mathbb{P}_{X|A}(x) = \begin{cases} \dfrac{\mathbb{P}_X(x)}{\mathbb{P}(A)} & \text{if } x \in A \\ 0 & \text{o.w.} \end{cases}$$

Ex:

$X \sim \text{Unif}[0,2]$

Let A be the event that $X \in [0,1]$

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}(A)} & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{\frac{1}{2}}{1/2} & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

Conditioning on an Event

Consider a continuous random variable X, and an event A.
Let $\mathcal{A}$ be the set of values $X(\omega)$ for all $\omega \in A$. So, $P(X \in \mathcal{A}) = P(A)$.

$$f_{X|A}(x)\delta = P(x \leq X \leq x + \delta | X \in \mathcal{A})$$

$$= \frac{P(x \leq X \leq x + \delta \cap X \in \mathcal{A})}{P(X \in \mathcal{A})} \quad = \begin{cases} \frac{f_X(x)\delta}{P(A)} & \forall x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)} & \forall x \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases}$$

Discrete vs Continuous Recap

Note: The pdf of a continuous random variable can be greater than 1 !

Consider $X \sim \text{Unif}[0, \frac{1}{2}]$

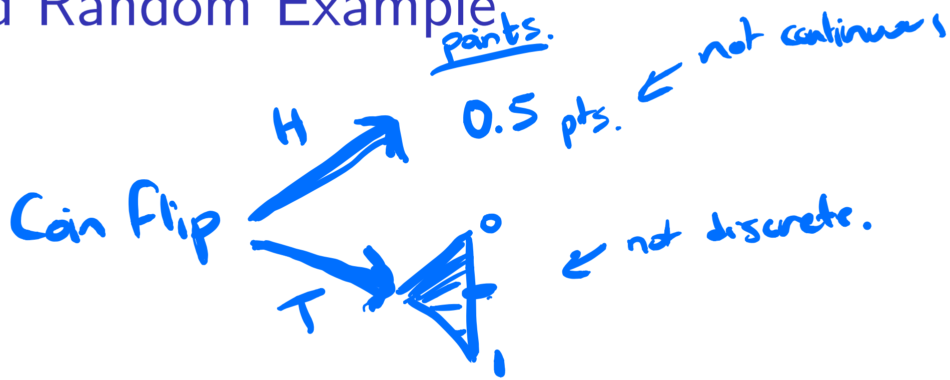
Discrete	Continuous
X	X
mass	density
PMF P(X=x)	PDF $f_X(x)$
CDF $P(X \leq x)$	CDF $P(X \leq x)$
$E[X] = \Sigma_x x P(X = x)$	$E[X] = \int_{-\infty}^{\infty} x \underbrace{f(x)}_{\text{pdf}} dx$

prob. $X \in [a, b]$

Mixed Random Variables

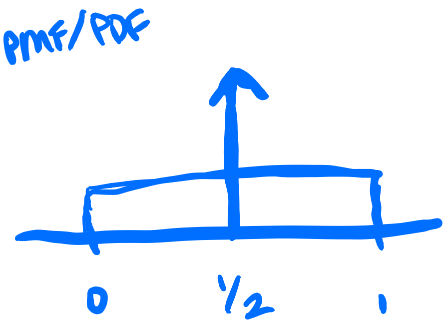
Some random variables are neither continuous nor discrete, but rather a combination of the two.

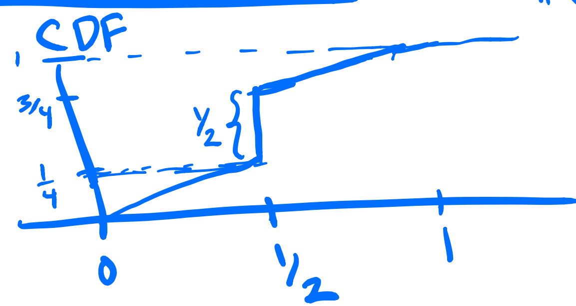
Mixed Random Example



You flip a fair coin. If it is heads, then you get a reward of 0.5 points. If it is tails, you spin a wheel to get a point value in [0, 1]. Let X be the mixed random variable representing the amount of points you have at the end of this experiment.

$\mathbb{P}(X \le x)$





Conditional Expectation

Let A be an event, and X be a continuous random variable. Then,

$$\mathbb{E}[X|A] = \int_{-\infty}^{\infty} x \cdot \underbrace{f_{X|A}(x)}_{\text{conditional pdf}}$$

This also holds in the discrete case, just use the conditional pmf instead of the conditional pdf.

Then we also have the conditional expectation version of the law of total probability:

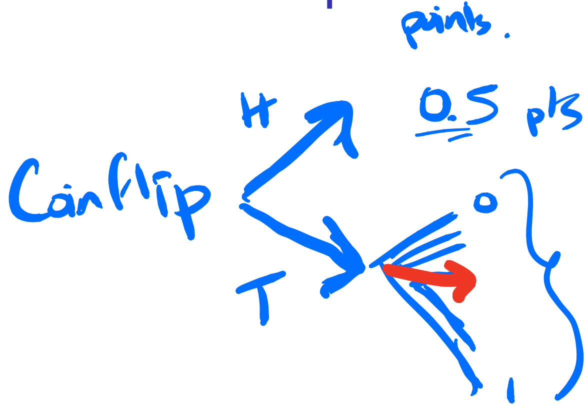
$$\mathbb{E}[X] = \mathbb{E}[X|A] \cdot P(A) + \mathbb{E}[X|A^c] \cdot P(A^c)$$

Exercise: Prove this

Hint: ① $\mathbb{E}[X] = \sum_x x \, \mathbb{P}(X=x)$

② Use law of total prob. on $\mathbb{P}(X=x)$

Conditional Expectation Example



$$\mathbb{E}[X \mid T] = \int_{-\infty}^{\infty} x \cdot \underbrace{p_{X|T}(x)}_{\text{pdf of unif}(0,1)}\, dx$$

$$= 0.5$$

What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = \mathbb{E}[X \mid H] \cdot \mathbb{P}(H) + \underbrace{\mathbb{E}[X \mid T]} \cdot \mathbb{P}(T)$$

$$= (0.5) \cdot (0.5) + (0.5) \cdot (0.5)$$

$$= 0.25 + 0.25 = \underline{0.5}$$

Conditional Expectation Example

Consider the mixed random variable X from before. What is $\mathbb{E}[X]$? Let A be the event the coin lands on heads.

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X|A] \cdot P(A) + \mathbb{E}[X|A^c] \cdot P(A^c) && (1) \\ &= 0.5 \cdot 0.5 + 0.5 \cdot 0.5 && (2) \\ &= 0.5 && (3) \end{aligned}$$

Two Envelope Paradox

identical in appearance, weight etc.

One envelope has \$$x$, and the other has \$$2x$.

You get one of the envelopes at random.

You then get a chance to keep it (and the money inside)

or you can switch to other one.

Two Envelope Paradox

There are two envelopes. One of them has x dollars, and the other has $2x$. You are given one of the envelopes. Should you switch to the other envelope?
Argument 1: It doesn't matter; by symmetry.
Argument 2: Let A be the amount in the envelope you are given, and B be the amount in the other one.

$$\begin{aligned}
\mathbb{E}[B] &= \mathbb{E}[B|A < B] \cdot P(A < B) + \mathbb{E}[B|A > B] \cdot P(A > B) && (4)\\
&= \mathbb{E}[B|\boxed{B = 2A}] \cdot \frac{1}{2} + \mathbb{E}[B|B = \frac{A}{2}] \cdot \frac{1}{2} && (5)\\
&= \mathbb{E}[2A] \cdot \frac{1}{2} + \mathbb{E}[\frac{A}{2}] \cdot \frac{1}{2} && (6)\\
&= \frac{5}{4}\mathbb{E}[A] && (7)
\end{aligned}$$

Problem:
These A's are not the same:

Fixed
$= 2x \cdot \frac{1}{2} + x \cdot \frac{1}{2}$
$= x + \frac{1}{2}x = \boxed{1.5x}$

Also, $\mathbb{E}[A] = 1.5x$ so we're good.

so switch?