

## 1 Short Answers - Graphs

- (a) Bob removed a degree 3 node from an  $n$ -vertex tree. How many connected components are there in the resulting graph?
- (b) Given an  $n$ -vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph?

## 2 Planarity

- (a) Prove that  $K_{3,3}$  is nonplanar.
- (b) Consider graphs with the property  $T$ : For every three distinct vertices  $v_1, v_2, v_3$  of graph  $G$ , there are at least two edges among them. Use a proof by contradiction to show that if  $G$  is a graph on  $\geq 7$  vertices, and  $G$  has property  $T$ , then  $G$  is nonplanar.

## 3 Graph Coloring

Prove that a graph with maximum degree at most  $k$  is  $(k + 1)$ -colorable.

## 4 Hypercubes

The vertex set of the  $n$ -dimensional hypercube  $G = (V, E)$  is given by  $V = \{0, 1\}^n$  (recall that  $\{0, 1\}^n$  denotes the set of all  $n$ -bit strings). There is an edge between two vertices  $x$  and  $y$  if and only if  $x$  and  $y$  differ in exactly one bit position. These problems will help you understand hypercubes.

(a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Show that for any  $n \geq 1$ , the  $n$ -dimensional hypercube is bipartite.