

## 1 The Count

- (a) How many of the first 100 positive integers are divisible by 2, 3, or 5?
- (b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?
- (c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

## 2 CS70: The Musical

Edward, one of the previous head TA's, has been hard at work on his latest project, *CS70: The Musical*. It's now time for him to select a cast, crew, and directing team to help him make his dream a reality.

- (a) First, Edward would like to select directors for his musical. He has received applications from  $2n$  directors. Use this to provide a combinatorial argument that proves the following identity:  $\binom{2n}{2} = 2\binom{n}{2} + n^2$
- (b) Edward would now like to select a crew out of  $n$  people, Use this to provide a combinatorial argument that proves the following identity:  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  (this is called pascal's identity)
- (c) There are  $n$  actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity:  $\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$
- (d) Generalizing the previous part, provide a combinatorial argument that proves the following identity:  $\sum_{k=j}^n \binom{n}{k} \binom{k}{j} = 2^{n-j} \binom{n}{j}$ .

### 3 Bit String

How many bit strings of length 10 contain at least five consecutive 0's?