EECS 70 Discrete Mathematics and Probability Theory
Summer 2020

Note 4

1 Sets

A **set** is a well defined collection of objects. These objects are called **elements** or **members** of the set, and they can be anything, including numbers, letters, people, cities, and even other sets. By convention, sets are usually denoted by capital letters and can be described or defined by listing its elements and surrounding the list by curly braces. For example, we can describe the set A to be the set whose members are the first five prime numbers, or we can explicitly write: $A = \{2, 3, 5, 7, 11\}$. If x is an element of A, then we write $x \in A$. Similarly, if y is not an element of A, then we write $y \notin A$. Two sets A and B are said to be **equal**, written as $A = B$, if they have the same elements. The order and repetition of elements do not matter, so {red, white, blue} = {blue, white, red} = {red, white, white, blue}. Sometimes, more complicated sets can be defined by using a different notation. For example, the set of all rational numbers, denoted by $\mathbb{Q}$, can be written as: $\{\frac{a}{b} \mid a, b \text{ are integers}, b \neq 0\}$. In English, this is read as "the set of all fractions such that the numerator is an integer and the denominator is a non-zero integer."

Cardinality

We can also talk about the size of a set, or its **cardinality**. If $A = \{1,2,3,4\}$, then the cardinality of A, denoted by $|A|$, is 4. It is possible for the cardinality of a set to be 0. There is a unique such set, called the **empty set**, denoted by the symbol $\emptyset$. A set can also have an infinite number of elements, such as the set of all integers, prime numbers, or odd numbers.

Subsets and Proper Subsets

If every element of a set A is also in set B, then we say that A is a **subset** of B, written $A \subseteq B$. Equivalently we can write $B \supseteq A$, or B is a superset of A. A **proper subset** is a set A that is strictly contained in B, written as $A \subset B$, meaning that A excludes at least one element of B. For example, consider the set $B = \{1, 2, 3, 4, 5\}$. Then $\{1, 2, 3\}$ is both a subset and a proper subset of B, while $\{1, 2, 3, 4, 5\}$ is a subset but not a proper subset of B. Here are a few basic properties regarding subsets:

- The empty set, denote by $\{\}$ or $\emptyset$, is a proper subset of any nonempty set A: $\{\} \subset A$.
- The empty set is a subset of every set B: $\{\} \subseteq B$.
- Every set A is a subset of itself: $A \subseteq A$.

Intersections and Unions

The **intersection** of a set A with a set B, written as $A \cap B$, is the set containing all elements which are in both A and B. Two sets are said to be **disjoint** if $A \cap B = \emptyset$. The **union** of a set A with a set B, written as $A \cup B$, is the set of all elements which are in either A or B or both. For example, if A is the set of all positive even numbers, and B is the set of all positive odd numbers, then $A \cap B = \emptyset$, and $A \cup B = \mathbb{Z}^+$, or the set of all positive integers. Here are a few properties of intersections and unions:

- $A \cup B = B \cup A$

- $A \cup \emptyset = A$
- $A \cap B = B \cap A$
- $A \cap \emptyset = \emptyset$

Complements

If A and B are two sets, then the **relative complement** of A in B, or the **set difference** between B and A, written as $B - A$ or $B \backslash A$, is the set of elements in B, but not in A: $B \backslash A = \{x \in B \mid x \notin A\}$. For example, if $B = \{1,2,3\}$ and $A = \{3,4,5\}$, then $B \backslash A = \{1,2\}$. For another example, if $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, then $\mathbb{R} \backslash \mathbb{Q}$ is the set of irrational numbers. Here are some important properties of complements:

- $A \backslash A = \emptyset$
- $A \backslash \emptyset = A$
- $\emptyset \backslash A = \emptyset$

Significant Sets

In mathematics, some sets are referred to so commonly that they are denoted by special symbols. These include:

- $\mathbb{N}$ denotes the set of all natural numbers: $\{0,1,2,3,...\}$.
- $\mathbb{Z}$ denotes the set of all integer numbers: $\{\ldots,-2,-1,0,1,2,\ldots\}$.
- $\mathbb{Q}$ denotes the set of all rational numbers: $\{\frac{a}{b} \mid a,b \in \mathbb{Z},\, b \neq 0\}$.
- $\mathbb{R}$ denotes the set of all real numbers.
- $\mathbb{C}$ denotes the set of all complex numbers.

Products and Power Sets

The **Cartesian product** (also called the **cross product**) of two sets A and B, written as $A \times B$, is the set of all pairs whose first component is an element of A and whose second component is an element of B. In set notation, $A \times B = \{(a,b) \mid a \in A, b \in B\}$. For example, if $A = \{1,2,3\}$ and $B = \{u,v\}$, then $A \times B = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$. And $\mathbb{N} \times \mathbb{N} = \{(0,0),(1,0),(0,1),(1,1),(2,0),\ldots\}$ is the set of all pairs of natural numbers. Given a set S, the **power set** of S, denoted by $\mathscr{P}(S)$, is the set of all subsets of S: $\{T \mid T \subseteq S\}$. For example, if $S = \{1,2,3\}$, then the power set of S is: $\mathscr{P}(S) = \{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. Note that, if $|S| = k$, then $|\mathscr{P}(S)| = 2^k$. [Why?]

2 Bijections

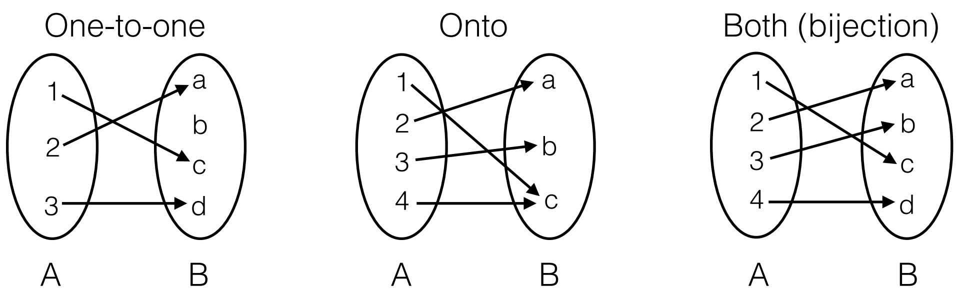
Two finite sets have the same size if and only if their elements can be paired up, so that each element of one set has a unique partner in the other set, and vice versa. We formalize this through the concept of a *bijection*.

Consider a function[1] (or mapping) f that maps elements of a set A (called the *domain* of f) to elements of set B (called the *range* of f). Since f is a function, it must specify, for each element $x \in A$ ("input"), exactly one element $f(x) \in B$ ("output"). Recall that we write this as $f : A \to B$. We say that f is a *bijection* if every element $a \in A$ has a unique *image* $b = f(a) \in B$, and every element $b \in B$ has a unique *pre-image* $a \in A$ such that $f(a) = b$.

f is a *one-to-one function* (or an *injection*) if f maps distinct inputs to distinct outputs. More rigorously, f is one-to-one if the following holds: $x \neq y \Rightarrow f(x) \neq f(y)$.

f is *onto* (or *surjective*) if it "hits" every element in the range (i.e., each element in the range has at least one pre-image). More precisely, a function f is onto if the following holds: $(\forall y\ \exists x)(f(x) = y)$.

Here are some simple examples to help visualize one-to-one and onto functions, and bijections:



Note that, according to the above definitions, f is a bijection if and only if it is both one-to-one and onto.

3 Sequences

A sequence is a discrete structure used to represent an ordered list (and thus plays an important role in computer science). For example, $2, 1, 3$ is a sequence with three terms, and $1, 1, 2, 3, 5, 8, \cdots$ is a sequence of infinite terms. Formally, a **sequence** is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \dots\}$ or the set $\{1, 2, 3, \dots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call an a term of the sequence.

There is a compact notation for writing sums or products a sequence. For example, to write $1 + 2 + \cdots + n$, without having to say "dot dot dot", we can write $\sum_{i=1}^{n} i$.

More generally we can write the sum $f(m) + f(m+1) + \cdots + f(n)$ as $\sum_{i=m}^{n} f(i)$. Thus, for example, $\sum_{i=5}^{20} i^2 = 5^2 + 6^2 + \cdots + 20^2$.

Analogously, to write the product $f(m)f(m+1)\cdots f(n)$ we use the notation $\prod_{i=m}^{n} f(i)$. As an example, $\prod_{i=1}^{n} i = 1 \cdot 2 \cdots n$ is the product of the first n positive integers.

[1] See an earlier note for a review of basic definitions connected with functions.

4 Exercise

1. Let $f : A \to B$ be a bijection. Show that f has an *inverse* $f^{-1} : B \to A$ that satisfies $f^{-1}(f(a)) = a$ for all $a \in A$, and that f^{-1} is also a bijection.