EECS 70 Discrete Mathematics and Probability Theory

Fall 2014 Anant Sahai

Midterm 1

Exam location: 10 Evans, Last name starting with A-B or R-T

PRINT your student ID: ____

PRINT AND SIGN your name: __________________, __________________ ______________________

(last) (first) (signature)

PRINT your Unix account login: cs70-_____________

PRINT your discussion section and GSI (the one you attend): ____

Name of the person to your left: ____

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Name of the person in front of you: ____

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Section 0: Pre-exam questions (3 points)

1. **What other courses are you taking this term? (1 pt)**

2. **What activity do you really enjoy? Describe how it makes you feel. (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

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Section 1: Straightforward questions (24 points)

Unless told otherwise, you must show work to get credit. You get one drop: do 4 out of the following 5 questions. (We will grade all 5 and keep only the best 4 scores) However, there will be essentially no partial credit given in this section. Students who get all 5 questions correct will receive some bonus points.

3. XOR

The truth table of XOR is as follows.

A	B	A XOR B
F	F	F
F	T	T
T	F	T
T	T	F

(a) **Express XOR using only** $(\wedge, \vee, \neg)$ **and parentheses.**

These are all correct:

A XOR $B = (A \wedge \neg B) \vee (\neg A \wedge B)$

A XOR $B = (A \vee B) \wedge (\neg A \vee \neg B)$

A XOR $B = (A \vee B) \wedge \neg(A \wedge B)$

(b) **Does** $(A$ **XOR** $B)$ **imply** $(A \vee B)$**? Explain briefly.**

Yes. $(A \text{ XOR } B) = T \implies ((A = T) \wedge (B = F)) \vee ((A = F) \wedge (B = T)) \implies (A \vee B = T)$.

(c) **Does** $(A \vee B)$ **imply** $(A$ **XOR** $B)$**? Explain briefly.**

No. When $(A = T) \wedge (B = T)$, then $(A \vee B) = T$ but $(A \text{ XOR } B) = F$.

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4. Stable Marriage

Consider the set of men $M = \{1, 2, 3\}$ and the set of women $W = \{A, B, C\}$ with the following preferences.

Men	Women		
1	A	B	C
2	B	A	C
3	A	B	C

Women	Men		
A	2	1	3
B	1	2	3
C	1	2	3

Run the male propose-and-reject algorithm on this example. How many days does it take and what is the resulting pairing? (Show your work)

The algorithm takes 3 days to produce a matching. The resulting pairing is {(A, 1), (B, 2), (C, 3)}

Woman	Day 1	Day 2	Day 3
A	(1),3	(1)	(1)
B	(2)	(2),3	(2)
C			(3)

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5. Prove it

Suppose x, y are integers. **Prove that if 5 does not divide xy, then 5 does not divide x and 5 does not divide y.**

We will use proof by contraposition.

Original statement: $5 \nmid xy \to (5 \nmid x \wedge 5 \nmid y)$

Contraposition: $\neg(5 \nmid x \wedge 5 \nmid y) \to 5 \nmid xy$.

By De Morgan's laws, $\neg(5 \nmid x \wedge 5 \nmid y) = 5|x \vee 5|y$, and the contraposition becomes $5|x \vee 5|y \to 5|xy$.

We consider the following three possibilities.

Case 1. Suppose $5|x$, but $5 \nmid y$. Then $x = 5a$ for some $a \in \mathbb{Z}$. From this we get $xy = 5(ay)$, and that means $5|xy$.

Case 2. Suppose $5|y$, but $5 \nmid x$. Then $y = 5a$ for some $a \in \mathbb{Z}$. From this we get $xy = 5(ax)$, and that means $5|xy$.

Case 3. Suppose $5|x$ and $5|y$. Then $x = 5a$ and $y = 5b$ for some $a, b \in \mathbb{Z}$. From this we get $xy = (5a)(5b) = 5(5ab)$, and that also means $5|xy$.

The above cases show that if $5|x$ or $5|y$, then $5|xy$.

Since the contraposition of the original statement holds, the original statement must also be true.

6. Inequality

Prove *by induction* on n that if n is a natural number and $x > 0$, then $(1+x)^n \geq 1+nx$.

We will prove the claim by induction on n.

- *Base case*: When $n = 0$ the claim holds since $(1+x)^0 \geq 1+0x$.
- *Inductive hypothesis*: Now, assume as our inductive hypothesis that $(1+x)^k \geq 1+kx$ for some value of $n = k$ where $k > 0$.
- *Inductive step*: For $n = k+1$, we can show the following chain of inequalities:

$$\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) \\
&\geq (1+kx)(1+x) \quad \text{(by the inductive hypothesis)} \\
&\geq 1+kx+x+kx^2 \\
&\geq 1+(k+1)x+kx^2 \\
&\geq 1+(k+1)x \quad (kx^2 > 0 \text{ since } k > 0, x > 0)
\end{aligned}$$

  By induction, we have shown that $\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx$.

Common mistakes:

- A lot of students tried to work backwards from what needs to be proven. They started with $(1+x)^{n+1} \geq 1+(n+1)x$ and proceeded to manipulate both sides of the equation in the inductive step. This is not a proper induction proof. You need to start on the left hand side and use the inductive hypothesis to get to the right hand side.
- Unless otherwise noted, you can always assume that natural numbers start from 0 in this class. A few students use $n = 1$ as their base case.
- Some students induct on the wrong variable (x).
- A few forgot to mention how they dropped nx^2 in the last step. Points were not deducted in this exam, but please make sure to make your proof bulletproof in the future regardless of how obvious the explanation can be.

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7. Stable Marriage

Below are the observed proposals from the traditional male propose-and-reject algorithm.

Day	Women	Men	Day	Women	Men
1	A	-	5	A	-
	B	1, 2		B	4
	C	3		C	2
	D	4		D	1, 3
2	A	-	6	A	-
	B	1		B	3, 4
	C	2, 3		C	2
	D	4		D	1
3	A	-	7	A	-
	B	1		B	3
	C	2		C	2, 4
	D	3, 4		D	1
4	A	-	8	A	2
	B	1, 4		B	3
	C	2		C	4
	D	3		D	1

After the algorithm terminates, only B is married to the man she likes the most. Also, it is known that A has the same preferences as B and every man likes C better than A.

Reconstruct the complete preference lists of men and women given the information above. (You do not have to show work.)

Men	Preferences	Women	Preferences
1	> > >	A	> > >
2	> > >	B	> > >
3	> > >	C	> > >
4	> > >	D	> > >

We know men propose starting from the top of their lists, therefore we can fill their preference lists with the series of women they proposed to. We also know that if a woman receives multiple proposals, she tells the man she likes the most to comeback the next day. So, if men 1 and 2 proposes to the same woman on one day, and 1 comes back to her the next day but 2 doesn't, it means the woman prefers 1 over 2. These reasonings give us the partially filled preference lists.

Men	Preferences	Women	Preferences
1	$B > D > ? > ?$	A	$? > ? > ? > ?$
2	$B > C > A > ?$	B	$3 > 4 > 1 > 2$
3	$C > D > B > ?$	C	$4 > 2 > 3$
4	$D > B > C > ?$	D	$1 > 3 > 4$

By process of elimination we know who are in 2, 3, and 4's last spots. C and A are both missing from 1's list, but since every man likes C better than A, 1 must too. For women, A has the same preferences as B so we can just copy B's list to A. 1 and 2 are missing from C and D's lists, respectively, and unlike men's preferences, these missing men can be anywhere in the women's lists. Fortunately, B being the only woman that is married to her top man means C and D must not like 4 and 1, the men they are married to, the most, and we can put the missing men in their top spots.

Men	Preferences
1	$B > D > C > A$
2	$B > C > A > D$
3	$C > D > B > A$
4	$D > B > C > A$

Women	Preferences
A	$3 > 4 > 1 > 2$
B	$3 > 4 > 1 > 2$
C	$1 > 4 > 2 > 3$
D	$2 > 1 > 3 > 4$

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Section 2: True/False (30 points)

For the questions in this section, determine whether the statement is true or false. If true, prove the statement is true. If false, provide a counterexample demonstrating that it is false.

You get one drop: do 2 out of the following 3 questions. (We will grade all three questions and keep only the best two scores.) Students who get all three questions perfectly correct will receive some bonus points.

8. Sets (15 points)

If $n > 0$ is a positive integer, and S is a set of distinct positive integers, all of which are less than or equal to n, then S has at most n elements.

Mark one: TRUE or FALSE.

The implication is TRUE

1. **Proof by Induction**
   Base Case: For $n = 1$, S cannot contain any element > 1. Therefore S can only be: $\emptyset$, or $\{1\}$. In both cases, is true that $|S| \leq n = 1$.
   Inductive Hypothesis: Assume that for $n = k$, **all sets** S for which $\forall x \in S : 1 \leq x \leq k$, satisfy $|S| \leq k$.
   Induction Step: For $n = k+1$, consider a set S such that $\forall x \in S : 1 \leq x \leq k+1$. Consider partitioning S into two sets A and B, such that $A = \{x \in S : 1 \leq x \leq k\}$ and $B = \{x \in S : x > k\}$. By the hypothesis, $|A| \leq k$. Further, by definition we have $\forall x \in B : k < x \leq k+1$. Thus B cannot contain any element $x \in \mathbb{Z}, x \neq k+1$. So B can only be $\emptyset$ or $\{k+1\}$. In both cases, $|B| \leq 1$. Since A and B partition S, we have $|S| = |A| + |B| \leq k+1$.
2. **Proof by Contradiction**
   Suppose there exists a set S such that $\forall x \in S : 1 \leq x \leq n$ but $|S| > n$. Apply the Pigeonhole Principle: There are n positive integers between 1 and n (boxes), and each distinct $x \in S$ (pigeons) must be placed in one of these boxes. By the Pigeonhole Principle, there must exist some two pigeons in the same box – or some two elements x_1, x_2 that are not distinct. Contradiction.
3. **Direct Proof**
   Let $A_n = \{x \in \mathbb{Z} : 1 \leq x \leq n\}$. By explicit enumeration, A_n contains exactly n distinct elements, so $|A_n| = n$. Then any set S such that $\forall x \in S : 1 \leq x \leq n$ is a subset of A_n. And $S \subseteq A_n \implies |S| \leq |A_n| = n$.

All of the above proofs were accepted, though the first is most rigorous. In fact, the proving the Pigeonhole Principle, and proving (rigorously) that $A \subseteq B \implies |A| \leq |B|$ requires a very similar inductive argument.
This problem statement is so intuitively obvious that many people accidentally assumed the statement in their proof. For example, in the inductive step we cannot say something like "remove the maximum element of S", since technically we do not know that S is finite (without assuming the statement itself). We did not require such levels of rigor in grading this problem, but such things are good to keep in mind.

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9. Does "No" Matter? (15 points)

Consider an alternative to the Propose and Reject algorithm (with no rejections), where women take turns choosing the best available husband from the remaining unchosen men. On day 1, the oldest woman chooses her most preferred man, and marries him. On day k, the k-th eldest woman chooses her most preferred choice from the remaining unmarried men, and marries him. **No matter what the preferences are, this process always results in a stable matching.**

Mark one: TRUE or FALSE.

The statement is FALSE

Consider the set of men $M = \{1,2\}$ and women $W = \{A,B\}$ with the following preferences.

Men	Women
1	B > A
2	A > B

Women	Men
A	1 > 2
B	1 > 2

Running the alternative Propose and Reject algorithm on this set yields the pairings: $(A,1),(B,2)$. However, we can see that Man 1 prefers Woman B and vice versa, forming a rogue pair. Therefore, the pairing is not stable. We found a counterexample, so the statement is FALSE

10. *n* Matchings (15 points)

For all positive n, it is always possible to construct a set of preferences for n women and n men such that at least n distinct stable matchings are possible.

Mark one: TRUE or FALSE.

The statement is TRUE

Consider preference lists where man m's j-th preference (zero-based) is woman $(m+j) \bmod n$, and woman w's j-th preference (zero-based) is man $(w+j+1) \bmod n$, $\forall m,w,j \in \mathbb{Z}, 0 \le m,w,j < n$. For example, the preference lists for $n = 5$ are shown below.

Men	Preferences
0	$0 > 1 > 2 > 3 > 4$
1	$1 > 2 > 3 > 4 > 0$
2	$2 > 3 > 4 > 0 > 1$
3	$3 > 4 > 0 > 1 > 2$
4	$4 > 0 > 1 > 2 > 3$

Women	Preferences
0	$1 > 2 > 3 > 4 > 0$
1	$2 > 3 > 4 > 0 > 1$
2	$3 > 4 > 0 > 1 > 2$
3	$4 > 0 > 1 > 2 > 3$
4	$0 > 1 > 2 > 3 > 4$

Claim: A matching where all men are paired with the j-th women in their preference lists is stable.

Proof: We will call this matching M_j. Notice that all men are matched with their j-th woman, and all women are matched with their $(n-j-1)$-th man. Formally, we must show that there cannot be a rogue couple. From the way men's preferences are generated, we can find which man m has woman w in his j-th spot,

$$\begin{aligned}(m+j) \bmod n &= w \\ m &= (w-j) \bmod n. \end{aligned} \tag{1}$$

From this, we find a set R_w of men who could form a rogue couple with a woman w, $0 \le w < n$. They must prefer w over their partners in M_j. In other words, they must put w in a spot $k < j$.

$$R_w = \{m' \in \mathbb{Z} \mid m' = (w-k) \bmod n, \forall k \in \mathbb{Z}, 0 \le k < j\}. \tag{2}$$

Similarly, we work on the women's preference formula to find the rank r of a man m in woman w's list.

$$\begin{aligned}(w+r+1) \bmod n &= m \\ r &= (m-w-1) \bmod n \end{aligned} \tag{3}$$

Now, we substitute m in Equation (3) with $(w-k) \bmod n$ to find out how woman w ranks each $m' \in R_w$,

$$\begin{aligned} r_{m'} &= (((w-k) \bmod n) - w - 1) \bmod n \\ r_{m'} &= (w-k-w-1) \bmod n = (-k-1) \bmod n \\ r_{m'} &= n-1-k \qquad (0 \le k < j < n) \end{aligned} \tag{4}$$

Substituting m in Equation (3) with $(w-j) \bmod n$, we get the rank of w's current partner in her preference list,

$$\begin{aligned} r_m &= (((w-j) \bmod n) - w - 1) \bmod n \\ r_m &= n-1-j \end{aligned} \tag{5}$$

Since $r_m = n-1-j < r_{m'} = n-1-k, \forall k \in \mathbb{Z}, 0 \le k < j$, woman w likes her partner better than all men who like her better than their partners. Therefore, a rogue couple cannot exists, and the matching M_j is stable.

Because there are n distinct j's in range $0 \le j < n$, the proposed preference lists have at least n possible stable matchings. □

Alternatively, one could extend the Pairing Up example discussed in Discussion 3W, since it is guaranteed to have at least $2^{\lfloor n/2 \rfloor}$ stable matchings, which is greater than or equal to n for all $n \ge 2$. To generalize the method to support odd n, it is sufficient to give preference lists for $n = 2$ and $n = 3$ and prove that we can get to higher n's by dividing a person to a block of 2×2 anti-soulmates using strong induction on n. In this case, there will be at least $2^{\lfloor n/2 \rfloor}$ and $2^{\lfloor n/2 \rfloor} + 1$ matchings for even and odd n, respectively.

$n = 2$

Men	Preferences
0	$0 > 1$
1	$1 > 0$

Women	Preferences
0	$1 > 0$
1	$0 > 1$

$n = 3$

Men	Preferences
0	$0 > 1 > 2$
1	$1 > 0 > 2$
2	$2 > 0 > 1$

Women	Preferences
0	$1 > 0 > 2$
1	$0 > 1 > 2$
2	$2 > 0 > 1$

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Section 3: Free-form Problems (65 points)

11. Bieber Fever (35 points)

In this world, there are only two kinds of people: people who love Justin Bieber, and people who hate him. We are searching for a stable matching for everyone. The situation is as follows:

- For some $n \geq 5$, there are n men, n women, and one Justin Bieber[1].
- Men can be matched with women; or anyone can be matched with Justin Bieber.
- Everyone is either a Hater or a Belieber. Haters want to be matched with anyone but Justin Bieber. Beliebers really want to be matched with Justin Bieber but don't mind being matched with other people.
- Men and women still have preference lists, as usual, but if they are a Belieber, Justin Bieber is always in the first position. If they are a hater, Justin Bieber is always in the last position.
- Justin Bieber desires to have 10 individuals matched with him (to party forever). As Justin Bieber is a kind person and wishes to be inclusive, he wishes to have exactly 5 women and 5 men in his elite club.
- Justin Bieber also has a preference list containing all $2n$ men and women.

A stable matching is defined as follows:

- Justin Bieber has 10 partners, of which 5 are men and 5 are women.
- All men and women not matched up with Justin Bieber are married to someone of the opposite gender.
- No rogue couples exist; i.e., there is no man M and woman W such that M prefers W to his current wife, and W prefers M to her current husband.
- No Hater is matched with Justin Bieber.
- There is no man who (1) is not matched with Justin Bieber; and (2) who is preferred by Justin Bieber over one of his current male partners; and (3) who prefers Justin Bieber over his wife. And similarly for women vis-a-vis Justin Bieber relative to their husbands and Justin Bieber's female partners.

(a) (5 points) **Show that there does not necessarily exist a stable matching.**
Suppose that there are 5 men and 5 women, all haters. Then Justin Bieber certainly cannot be matched with 5 men and 5 women without being matched with a hater. Therefore a stable matching does not exist.

[1] For the purposes of this problem, Justin Bieber is neither male nor female.

(b) (10 points) **Provide an "if-and-only-if" condition for whether a stable matching exists.** (*No need to prove anything in this part. That comes in later parts of this question.*)

A stable matching exists if and only if there are at least 5 male Beliebers and at least 5 female Beliebers.

(c) (5 points) **Is Justin Bieber guaranteed to always get his Bieber-optimal group if a stable matching exists?** (Bieber-optimal means that he gets the best possible group that could be matched to him in any stable matching.)

Yes. Consider the group which consists of Justin Bieber's favorite 5 male Beliebers, and favorite 5 female Beliebers. If a stable matching exists, Justin Bieber must be matched with this group. Suppose towards a contradiction that he is matched with some other set of 5 male Beliebers. Then he is matched with some M who is not in his top 5 (i.e. M is ranked 6th or worse in Justin Bieber's preferences among male Beliebers), and he is not matched with some M^* who is in his top 5. But Justin prefers M^* to M, and M^* is a Belieber, violating condition 5. Thus this matching is not stable. The same reasoning follows symmetrically for females; therefore, there is only one group that Justin Bieber can possibly be matched with, and thus any stable matching is Justin Bieber-optimal.

(d) (10 points) **Give an algorithm which finds a stable matching if the condition you gave in (b) holds. Argue why this algorithm works.**

First, take Justin Bieber's 5 favorite male Beliebers and his 5 favorite female Beliebers, and match them with him. For example, suppose that Justin Bieber's preference list is A B 1 2 C D E F 3 4 5 G 6 H 7 I 8 9, and that A, B, D, E, G, H, I, 1, 3, 4, 7, 8, and 9 are Beliebers. Then we would match Justin Bieber with A, B, D, E, G and 1, 3, 4, 7, 8. Remove Justin Bieber and his crew from all remaining preference lists, and run the propose and reject algorithm on everyone else ($n-5$ men and $n-5$ women). Note that Justin Bieber's "favorite 5 male Beliebers" must exist, because there are at least 5 male Beliebers, and similarly for females. Now let us check the conditions. Certainly Justin Bieber is matched with 5 men and 5 women, none of whom are Haters, by construction (conditions 1 and 4). Because we matched Justin Bieber with his 5 favorite male Beliebers, there is no Belieber who is not matched with Justin Bieber who Justin Bieber prefers over one of his current partners (condition 5), and similarly for women. Finally, because we ran the propose-and-reject algorithm on the remaining people, we are guaranteed to have a matching with no rogue couples (conditions 2 and 3). Note that the definition of rogue couple can only apply to people who are not matched with Justin Bieber, so we don't need to concern ourselves with rogue couples involving anyone in Justin Bieber's crew.

For this problem, we saw several examples of incorrect algorithms. Algorithms that had two stages (first Justin Bieber collecting his crew, and then the rest running the propose-and-reject algorithm) generally worked. The algorithms which tried to run the propose-and-reject algorithm on Justin Bieber and everyone else "in parallel" were often incorrect in subtle ways. We also saw many algorithms which were not precisely stated. For example, students had men proposing to Justin Bieber without specifying how Justin Bieber should process those proposals, or other things like this.

In addition, although many students gave correct algorithms, many students did not give adequate arguments for stability. In particular, trying to prove stability for a single-stage algorithm is quite involved, because the old proofs no longer apply (one would essentially have to prove everything from scratch), and very few students gave correct arguments for such an algorithm. There were also other errors; for example, some students tried to use part c) to prove that Justin Bieber always gets his optimal group, but this is circular reasoning (this is a stable matching, therefore Justin Bieber gets his optimal group, therefore this is a stable matching).

(e) (5 points) **Prove that a stable matching cannot exist if the condition you gave in (b) does not hold.**
Suppose, without loss of generality, that there are less than 5 male Beliebers. Then, if Justin Bieber is matched with 5 males, at least one must be a hater, so the matching is unstable; and otherwise, Justin Bieber is not matched with 5 males, so the matching is also unstable. Therefore a stable matching does not exist.

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12. Tournament (30 points)

A Round Robin Tournament (RRT) between n people $A_1,\ldots,A_n$ is a tournament where every person plays every other person exactly once, and games never end in a tie. For example, an RRT between 3 people could result in

$$A_1 \to A_2$$
$$A_1 \to A_3$$
$$A_2 \leftarrow A_3$$

which is to say A_1 beat A_2 and A_3, and A_3 beat A_2. Suppose you only know how many wins everyone had. That is, person A_i only tells you W_i, their total number of wins.

(a) (5 points) **Show that such a tournament can have at most 1 person with 0 wins.**
Prove by contradiction. Suppose there were 2 or more people with 0 wins. Let two of those people be A_i, A_j. Then, A_i must have lost to everybody else, and A_j must have lost to everybody else as well. The match between A_i and A_j can have no winner, which is impossible, so by contradiction at most one person can have 0 wins.

(b) (5 points) **Show that such a tournament can have at most 1 person with $n-1$ wins.**
We can use a similar proof as part a. Suppose for contradiction that 2 or more people have $n-1$ wins. Let two of those be A_i, A_j. Since both A_i and A_j must win against every other person, the match between A_i and A_j must have two winners, which is impossible, so by contradiction at most one person can have $n-1$ wins.

(c) (10 points) **Prove that if each of the W_i's is unique, you can tell exactly who beat who.**
(HINT: It might be helpful to work out examples for $n = 2,3,4$ to help you see what is going on.)
Prove this by induction. First, note the following helpful lemma.

Lemma: There is a player with $n-1$ wins.

Proof (sketch): A player can win at most $n-1$ times and at least 0 times, since each person plays $n-1$ matches. This gives n possible values for each W_i, a number from 0 to $n-1$. If all W_i are unique, then each of these values must appear exactly once, and one of the W_i must equal $n-1$. (This is proved formally by induction as in the "Friends" problem on Discussion 1W, but we didn't require formal proof of this lemma on this exam). Now proceed by induction:

Base case: When $n = 2$, there is only one match, and we know exactly who wins that match - the player who reports a W_i of 1.

Inductive Hypothesis: If every W_i in a k-person round robin tournament is unique, we can find out the result of every match.

Inductive Step: Consider a $k+1$ person round robin tournament. By the lemma, there is a player A_j with $W_j = k$ wins. This person beat every other player, so we know the result of every match that involves him/her. Imagine the tournament in two stages: first player A_j beats everyone, then the remaining players play an RRT among each other. This second stage is a k-person RRT, in which all remaining players have $\widetilde{W_i} = W_i - 1$ wins among each other. Additionally, every $\widetilde{W_i}$ in this k-person RRT is unique, because the W_i from the original tournament were all unique. By the inductive hypothesis, we can find the result of every match that does not involve player A_j, and thus find the result of every match in the $k+1$-person tournament.

So, by induction, if each W_i is unique, we can find exactly who beat who**m**.

Remarks: Many students stated the lemma, then tried to proceed directly by saying "There must be someone who beat $n-1$ people, so he beat everyone. And there must be someone else who beat $n-2$ people, so he beat everyone except the first guy, ... etc"
This is entirely the right idea, and probably the most intuitive way to think about the problem. However, we only awarded partial credit for such approaches, since the "..." should be formalized by induction.

(d) (10 points) **Prove that if you can tell exactly who beat who, then each of the W_i's is unique.**

We will prove the contrapositive: If the W_i's are not all unique, then we can't tell exactly who beat whom.

Remark: At this point, many students had the right idea: "If all we know about players are their W_i, then obviously if two W_i's are the same we can't tell the two players apart." This is the right line of thinking, and we awarded partial to full credit depending on the level of justification given. Below we will give the rigorous proof, as a good example of how such ideas can be formalized.

First, we need to formalize what it means that "we can't tell" who beat whom. It's not enough to prove that one particular method of determining who-beat-whom (such as the one from part (c)) fails if some $W_i = W_j$ (many of you tried this). We need to show that **every possible method** of determining who-beat-whom must fail – in other words, we must show that if some $W_i = W_j$, then there does not exist a unique RRT with the given W_is.

Given a set of W_is such that $W_i = W_j$, consider the RRT R that generated the W_is (one such tournament exists, by the premise). We will construct a different RRT $\widetilde{R}$ that produces the same W_is.

Let (A_p, A_q) denote a match in which A_p beats A_q. Construct $\widetilde{R}$ consisting of exactly the matches in R, except with player A_i swapped with A_j in all matches involving them. For example (A_i, A_q) in the original tournament R becomes (A_j, A_q) in $\widetilde{R}$. Notice that the $\widetilde{W}_i$s produced from $\widetilde{R}$ are exactly the same as W_is from R, because by construction, all players beat the same number of people in both cases (though the people they beat may differ). It remains to show that R and $\widetilde{R}$ are different tournaments: Simply consider the result of the match between A_i and A_j – which must be different in both cases, by construction.

We have shown that if the W_is are not unique, then there exist at least two different tournaments which could have produced the W_is. Therefore it is not possible to uniquely determine who-beat-whom given non-unique W_is.

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13. (optional) Matchmaking Cruise (20 points)

Mr. and Mrs. Matchmaker are sponsoring a series of matchmaking cruises for single women. There are n women and n men with preference lists for each other. Assume n is even. The Matchmakers guarantee a spot on the ship for all n women, but can only fit $n/2$ men at a time. Since space is limited, the Matchmakers decide to let all n women aboard but the men are divided into two groups of $n/2$ men, Group A and Group B.

For the first "week," men from Group A are allowed to come aboard and start proposing to the n women through the male Propose and Reject algorithm, until no man receives any more rejections. (Just assume that a "week" is long enough for this to happen.) For the second week, Group A leaves and Group B starts proposing until no rejections are received. On the third week, Group A returns and this process continues to repeat until no man from either Group A or Group B is rejected.

During this process, if a woman still had an active proposal in hand from man a from Group A at the end of a particular week, then the next week, she will reject man b from Group B if she prefers a over b. On the other hand, if she prefers b over a, she will say "maybe" to b and reject a when he returns to the cruise the next week and re-proposes to her.

State and prove an Improvement Lemma for this scenario.

(This Improvement Lemma should be sufficiently powerful to be able to be used to eventually get a proof that the Matchmakers will end up with a stable pairing using their cruises.)

Improvement Lemma: If M proposes to W on the k-th day, then on every subsequent day she has someone M′ on a string whom she likes at least as much as M. The definition of W having M′ "on a string" is as follows: M′ is the latest man whom W said "maybe" to.

Proof: By the well-ordering principle, suppose towards a contradiction that the j-th day for $j > k$ is the first counterexample where W has either nobody or some M* inferior to M on a string. On day $j-1$, she has M′ on a string and likes M′ at least as much as M. According to the algorithm, we will have two cases:

1. On the j-th day, M′ is of the group proposing to W at that week. In this case, M′ will still propose to W on the j-th day. On the j-th day, W's best choice is at least as good as M′, and according to the algorithm, she will choose him over M*. This contradicts our initial assumption.
2. On the j-th day, M′ is not of the group proposing to W at that week. According to the algorithm, W will say "maybe" to somebody else on the j-th day only if W prefers that person over M′. Therefore if W has M* on the string on the j-th day, that means W prefers M* over M′ thus prefers M* over M. This contradicts our initial assumption. QED.

Weaker lemmas such as "If M proposes to W on the first day of the k-th week, then on the first day of every subsequent week she has someone M′ whom she likes at least as much as M on a string" is not strong enough to prove that the algorithm will end up with a stable pairing. It has to be combined with another improvement lemma within each week. We only gave partial credit to weaker improvement lemmas and proofs.

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[Doodle page! Draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]