CS 70	Discrete Mathematics and Probability Theory	
Fall 2016	Seshia and Walrand	Midterm 1

PRINT Your Name: ______________________, ______________________
(last) (first)

READ AND SIGN The Honor Code: *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.* ______________________

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EXAM VERSION: **A**

Name of the person sitting to your left: ______________________

Name of the person sitting to your right: ______________________

- After the exam starts, please *write your student ID (or name) on every page* (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- The questions vary in difficulty, so if you get stuck on any question, it might help to leave it and try another one.
- On questions 1-2: You need only give the answer in the format requested (e.g., true/false, an expression, a statement.) Note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 3-8, do give arguments, proofs or clear descriptions as requested.
- You may consult only *one sheet of notes*. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- There are **14** single sided pages on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 8 questions on this exam worth a total of 125 points.**

Do not turn this page until your instructor tells you to do so.

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1. TRUE or FALSE?: total 24 points, each part 3 points

For each of the questions below, answer TRUE or FALSE.

Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. $\forall x \exists y [P(x) \vee Q(y)]$ is equivalent to $[\forall x P(x)] \vee [\exists y Q(y)]$.

2. If P and Q are propositions, then $(P \vee Q) \Rightarrow (\neg Q)$ is always TRUE.

3. For the Stable Marriage Problem: A female-optimal pairing is male-pessimal.

4. In the Stable Marriage Algorithm (with men proposing), if W is last on every man's preference list and M is not last on any woman's preference list, M cannot end up paired with W.

5. The following statement is a proposition:
   "There is a unique integer solution to the equation $x^2 = 4$."

6. There exists a graph with 9 vertices, each of degree 3.

7. Consider an undirected graph G. If there is a (simple) path in G from vertex x to vertex y through vertex z, and there is a (simple) path in G from y to x through z, then there is a cycle in G containing x, y, and z.

8. If $x \equiv 5 \pmod 9$ and $y \equiv 4 \pmod 9$ then $x + y$ is divisible by 9.

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2. Short Answers: 5x3=15 points Clearly indicate your correctly formatted answer: this is what is to be graded.No need to justify!

1. Write the contrapositive of the following statement: If $x^2 - 3x + 2 = 0$, then $x = 1$ or $x = 2$

2. A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

3. An n-dimensional hypercube has 2^n vertices. How long can the shortest (simple) path between any two vertices in the hypercube be? (The length of a path is the number of edges in it.)

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4. TRUE or FALSE: Suppose you are given two trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ that share no vertices or edges. If you add an edge e connecting some vertex in V_1 to some vertex in V_2, then the resulting graph $(V_1 \cup V_2, E_1 \cup E_2 \cup \{e\})$ is also a tree.

5. TRUE or FALSE: In stable marriage, if Man M is at the top of Woman W's ranking but the bottom of every other woman's ranking, then every stable matching must pair M with W.

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3. Short Proofs: 4+4+4+4+4=20 points

1. Prove that $5\sqrt{2}$ is irrational.

2. A Pythagorean triple (a,b,c) has three natural numbers a,b,c such that $a^2+b^2=c^2$. Prove that at least one of a,b,c must be even.

3. Prove or disprove: If all vertices of an undirected graph have degree 4, the graph must be the complete graph on 5 vertices, K_5.

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4. Prove that for any integer n, if $n^3 + 2n + 3$ is odd, then n is even.

5. Recall that an *Eulerian walk* in an undirected graph G is a walk in G that traverses each edge exactly once.

   Consider n undirected graphs G_1, G_2, ... G_n that share no vertices or edges and have exactly two odd-degree vertices each. Prove that it is possible to construct an Eulerian tour visiting all of G_1, G_2, ... G_n using only n additional edges to connect them.

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4. Checking Proofs:3+3= 6 points

Each of the proofs below has a fallacy on a single line. Find the fallacy, and explain your answer briefly.

1. Proposition: For any integers x, y, and n, if $x-y$ is divisible by n, then so is $x+y$.

   Proof: If $x-y$ is divisible by n, then we can write $x-y \equiv 0 \pmod{n}$ or $x \equiv y \pmod{n}$.
   Squaring both sides, we get $x^2 \equiv y^2 \pmod{n}$.
   Taking square roots, we get $x \equiv -y \pmod{n}$.
   Rewriting, we get $x+y \equiv 0 \pmod{n}$, or $x+y$ is divisible by n. □

2. Proposition: Let a be a two digit (decimal) number and b be formed by reversing the digits of a. Then the digits of a^2 are simply those of b^2 reversed.
   (For example, if $a = 10$, $b = 01$, then $a^2 = 100, b^2 = 001$. Similarly, if $a = 12$,$b = 21$, we have $a^2 = 144, b^2 = 441$.)

   Proof: Let $a = 10x+y$ where x, y are decimal digits. Then $b = 10y+x$.
   This gives us:

$$a^2 = 100x^2 + 20xy + y^2 = 100x^2 + 10(2xy) + y^2$$
$$b^2 = 100y^2 + 20yx + x^2 = 100y^2 + 10(2yx) + x^2$$

   Thus, the digits of a^2 are x^2, $2xy$, and y^2 and similarly the digits of b^2 are y^2, $2yx$, and x^2, exactly reverse.
   This yields the desired result. □

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5. Proofs about XOR: 3+7= 10 points

Recall from Homework 1 the XOR operator, written $\oplus$: $P \oplus Q$ is TRUE if and only if exactly one of P and Q is TRUE and the other is FALSE.

1. Show that $\oplus$ is associative: given three propositions P_1, P_2, P_3, that $P_1 \oplus (P_2 \oplus P_3) \equiv (P_1 \oplus P_2) \oplus P3$.

2. Now, given n propositions P_1, P_2, ..., P_n, $n \geq 2$, we can construct the XOR of all of them: $P_1 \oplus P_2 \oplus P_3 \oplus \ldots \oplus P_n$. (Since $\oplus$ is associative, it does not matter how we put parentheses around them, so we omit this.) Call this Q_n; that is, $Q_n = P_1 \oplus P_2 \oplus P_3 \oplus \ldots \oplus P_n$.

   A *satisfying assignment* to Q_n is an assignment of TRUE/FALSE to the propositions $P_1, P_2, \ldots, P_n$ such that Q_n is TRUE. A *falsifying assignment* to Q_n is a TRUE/FALSE assignment to the P_is such that Q_n is FALSE.

   Prove that for all n, Q_n has exactly 2^{n-1} satisfying assignments.

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6. Perfect Matching in Graphs: 8+4+8=20 points

For an undirected (simple) graph with n vertices, where n is even, a *perfect matching* is a set of $n/2$ edges such that every vertex of the graph is incident to exactly one of the edges in the set.

1. Prove or disprove: Every tree has at most one perfect matching.

2. Prove or disprove: If a graph has a perfect matching, it is 2-colorable. (That is, each vertex can be assigned one of two colors so that no two adjacent vertices have the same color.)

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3. Prove that if G has the following property P:

   G is a simple graph with $2n$ ($n \geq 2$) vertices such that every vertex has degree $\geq n$

   then G has a perfect matching.
   (Hint: Prove that all graphs satisfying P have a Hamiltonian cycle; we suggest a proof by contradiction for this. Recall that a Hamiltonian cycle is one that visits each vertex exactly once.)

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7. Stable Marriage Problem: 3+7=10 points

Consider the following stable marriage instance.

Man	Women			
A	2	4	1	3
B	3	1	4	2
C	1	4	2	3
D	3	4	2	1

Woman	Men			
1	A	C	B	D
2	B	C	D	A
3	B	A	C	D
4	B	A	D	C

1. List all the rogue couples for the following pairing: (A,1), (B,2), (C,3), (D,4)

2. For each woman, find her optimal man and her pessimal man. Show all your work and justify your answer.

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8. Boolean Division: 10+10=20 points

Given predicates $F(x)$ and $D(x)$, we say that $D(x)$ is a *Boolean divisor* of $F(x)$ if there exist predicates $Q(x)$ and $R(x)$ such that $\forall x,\ F(x) = \{[D(x) \wedge Q(x)] \vee R(x)\}$, where $\exists x, \{D(x) \wedge Q(x) \neq \text{FALSE}\}$.
(In other words, a Boolean divisor is like integer division, where multiplication is replaced by AND, and addition by OR. Also note that we use "$=$" to mean propositional equivalence.)

A predicate $D(x)$ of $F(x)$ is said to be a *factor* of $F(x)$ if there exists a predicate $Q(x)$ such that $\forall x,\ F(x) = [D(x) \wedge Q(x)]$.

[Hint for both parts below: try using identities that simplify propositional forms.]

1. Prove that for any two predicates $F(x)$ and $D(x)$, $D(x)$ is a factor of $F(x)$ if and only if $\forall x, \{F(x) \wedge (\neg D(x)) = \text{FALSE}\}$.

2. Prove that for any two predicates $F(x)$ and $D(x)$, $D(x)$ is a Boolean divisor of $F(x)$ if and only if $\exists x, \{F(x) \wedge D(x) \neq \text{FALSE}\}$.

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