Midterm 1 Solutions

PRINT Your Name: Oski Bear

SIGN Your Name: $\mathscr{OSKI}$

PRINT Your Student ID: ____

CIRCLE your exam room:
Dwinelle 155 GPB 100 GPB 103 Soda 320 Soda 310 Cory 277 Cory 400 Other

Name of the person sitting to your left: Papa Bear

Name of the person sitting to your right: Mama Bear

- After the exam starts, please *write your student ID (or name) on every odd page* (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem unless we are clearly told in the space provided for the question to look elsewhere.
- On questions 1-2: You need only give the answer in the format requested (e.g., true/false, an expression, a statement.) We note that an expression may simply be a number or an expression with a relevant variable in it. **For short answer questions, correct clearly identified answers will receive full credit with no justification. Incorrect answers may receive partial credit.**
- On question 3-8, do give arguments, proofs or clear descriptions as requested.
- You may consult one sheet of notes. Apart from that, you may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- There are **14** single sided pages on the exam. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture.**
- **You have 120 minutes: there are 8 questions on this exam worth a total of 125 points.**

Do not turn this page until your instructor tells you to do so.

1. **TRUE or FALSE?: 2pts each**

For each of the questions below, answer TRUE or FALSE.

Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

Answer: Note that the answers provide explanations for your understanding, even though no such justification was required

1. $(\neg P \implies R) \land (\neg P \implies \neg R) \equiv P$

   Answer: True. This is proof by contradiction.
2. $\forall x \in \mathbb{N}, (P(x) \land (\exists y \in \mathbb{N}, Q(x,y)) \equiv \forall y \in \mathbb{N}, \exists x \in \mathbb{N}, P(x) \land Q(x,y)$.

   Answer: False. $P(x)$ is True. $Q(x,y)$ is $y > x$.
3. $(\neg P(0) \land \forall n \in \mathbb{N}, (P(n) \implies P(n-1))) \equiv \forall n \in \mathbb{N}, \neg P(n)$

   Answer: True. This is the well ordering principle on $\neg P(n)$.
4. $\forall x, ((P(x) \implies Q(x)) \land Q(x)) \equiv \forall x, P(x)$

   Answer: False. If $Q(x)$ is true that implies nothing about $P(x)$.
5. $P \lor Q \equiv \neg P \implies Q$ **Answer:** True. This is the logical equivalence of $P \implies Q$ and $\neg P \lor Q$.

   For the following two parts, assume that $Q(x,y)$ and $P(x)$ are predicates when $x, y \in \mathbb{N}$.
6. "$\forall x \in \mathbb{N}, P(x)$" is a proposition.

   Answer: TRUE. The forall x, eliminates the free variable, leaving a statement that is either true or false depending on whether $P(x)$ holds for every $x \in \mathbb{N}$.
7. "$\forall x \in \mathbb{N}, P(x) \land Q(x,y)$" is a proposition.

   Answer: False. It is a predicate where the free variable is y.
8. In a stable marriage instance where there is a man at the bottom of each woman's preference list, the man is paired with his least favorite woman in every stable pairing.

   Answer: False. Consider lists M1: W1>W2, M2: W2>W1, W1: M1>M2, W2: M1>M2.
9. In a stable marriage instance where there is a man at the top of each woman's preference list, the man is paired with his favorite woman in every stable pairing.

   Answer:

   True, if M is the man at the top and his favorite woman is W, then if (M,W') and (W,M') are in a pairing, (M,W) are a rogue couple because they mutually prefer each other.
10. Say I take a walk in any connected graph, making sure I only follow edges I haven't followed before, and keep walking until there are no edges I haven't followed incident to the the vertex I am in. If in the course of this walk I never see the same vertex twice then the degree of the vertex I get stuck at has degree exactly one.

    Answer: True. Since there is no cycle in the graph it is a simple path whose endpoints are degree 1.
11. For the stable roommates problem with $2n$ people, there is a stable pairing if every preference list has the form $\{p_1 > \cdots > p_{2n}\}$, that is, $i < j \implies p_i > p_j$.

    Answer: TRUE. The pairing $(1,2),(3,4),\ldots,(2n-1,2n)$ is stable.
12. A graph with k edges and n vertices has a vertex of degree at least $2k/n$.

    Answer: True. The sum of degrees is $2k$, there are n vertices. One is at least average.
13. If we remove one edge from K_n, the resulting graph can be vertex colored with $n-1$ colors.

    Answer: True. Remove a vertex with degree $n-2$. If one has $n-1$ colors available, one can color it. Henceforth, there is always a vertex of degree $n-2$.

14. For a graph with average degree k, more than half of the vertices must have degree at most k.

    Answer: False. Consider K_8 where every node has degree 7, then remove 6 edges from two vertices. 6 vertices have degree at least 5 since the form K_6, the number of edges is 16 corresponding to an average degree of 32/8 of 4.

15. If one adds an edge to a graph G and vertex colors it with 3 colors, then the original graph G is 3-colorable. **Answer:** True. The 3 coloring of the graph with an extra edge is a legal 3 coloring of G.

16. Any graph where $|E| \leq 3|V| - 6$ is planar.

    Answer: FALSE. The four dimensional hypercube is not planar. It has degree 4 or $|E| = 2|V|$ with $|V| = 16$, so $|E| = 32 \leq 3(16) - 6 = 42$. It is not planar as discussed in the notes.

17. For every connected, undirected graph there is a tour that uses every edge at least once and at most twice.

    Answer: True. Take a graph and make two copies of each edge. Now the graph has even degree and thus has an Eulerian tour that uses each edge once. This tour corresponds to using each original edge twice.

18. Any connected graph with average degree strictly less than 2 is a tree.

    Answer: True. It it is connected and has average degree less than 2 it must have $|V| - 1$ edges exactly. This means it is a tree.

19. Any walk in a hypercube that always traverses unused edges must return to the starting vertex before getting stuck at a vertex where all the edges have been used.

    Answer: False. For an n dimensional hypercube where n is odd, the degree is odd. In particular consider a $n = 1$ hypecube which is an edge.

20. Any complete graph has a Hamiltonian tour. (Recall that a Hamiltonian tour is a cycle that visits every **vertex** exactly once.)

    Answer: True. Any ordering of the vertices corresponds to a Hamiltonian tour as every edge is present.

21. Any graph where every triple of vertices is a triangle, i.e., for vertices $u,v,w \in V$, $(u,v),(v,w),(w,u) \in E$, is a complete graph.

    Answer: True. This condition implies every edge is present.

22. If $\gcd(x,y) = d$ and $gcd(y,z) = c$, then $gcd(x,z) \geq gcd(c,d)$.

    Answer: True. Any divisor of c and d are divisors of x and y.

23. The map $f(x) = ax \pmod{m}$ is a bijection from and to $\{0,\ldots,m-1\}$ when $gcd(a,m) = 1$. (A function $f : A \to B$ is a bijection from A to B if it is it is one-to-one, $f(x) \neq f(x')$ for $x \neq x'$, and for $y \in B$, there is a $x \in A$, where $f(x) = y$.)

    Answer: True. The existence of an inverse yields this statement.

2. An expression or number: 3 points each. Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

1. What is the minimum number of colors that are required to vertex color a graph that is a simple cycle of length k for $k \geq 2$? (**Answer is an expression possibly involving k.**)

   Answer: $2 + mod(k,2)$. For odd length cycles one needs 3 colors, for even one can use 2.

2. If $\gcd(x,y) = d$ then the smallest common multiple of x and y is?

   Answer: xy/d.

3. Let n be the largest number that K_n is a planar graph. **How many faces** does any planar drawing of this graph have? (**Answer is an expression or number.**)

   Answer: The number of vertices is 4, the number of edges is 6. The number of faces in a planar drawing is $e+2-v$ or 4.

4. Consider a planar graph G with e edges where each face is a triangle. How many vertices does it have? (**Answer is an expression or number.**)

   Answer: Euler's formula: $v+f=e+2$. Face edge incidences are $2e$ which equals $3f$. Plugging in, we obtain $v+\frac{2}{3}e=e+2$. Solving yields, $v=\frac{1}{3}e+2$.

5. How **many solutions** to $5x=25 \pmod{27}$? (**Answer is a number and recall that we are working modulo 27 so the maximum number of solutions is 27.**)

   Answer: 1. $gcd(5,27)=1$ so 5 has a multiplicative inverse modulo 27.

6. What day of the week is 10 years from today? (Note that there are two leap years between now and then.)

   Answer: Saturday. Thus, we get $365*10+2=5 \pmod 7$ days from today, which is Monday, so Saturday.

7. If we divide an n dimensional hypercube into two disjoint $n-1$ dimensional hypercubes, how many edges in the n dimensional hypercube have an endpoints in two different hypercubes. **In other words, how many edges "cross" between the subcubes.** (**Answer is an expression.**)

   Answer: 2^{n-1}. Each vertex in one cube is connected to the corresponding vertex in the other one.

8. Consider the graph G formed with vertex set $V=\{0,\ldots,m-1\}$, and edge set $E=\{(x,y): y=x+g \pmod m\}$ with $g\in\{0,\ldots,m/2-1\}$. What is the maximum length cycle in this graph? (**Answer is an expression. It may involve g and m.**)

   Answer: $d=m/gcd(g,m)$.

9. Recall that any graph where $e>3v-6$ is non-planar. Given an example of a graph where $e\le 3v-6$ that is non-planar.

   Answer: Either $K_{3,3}$ or take an edge in K_5 and split it into two edges.

10. Consider an n vertex planar graph with no degree one or two vertices. What is the minimum number of edges in such a graph? (**Answer is an expression perhaps involving n.**)

    Answer: $\frac{3n+mod(n,2)}{2}$. If n is even, every vertex can have degree 3, which suggests that there are $3n/2$ edges. If n is odd at most $n-1$ vertices have degree 3 and one has to have degree 4. Thus, the total degree is $3n+1$ and the total number of edges is $(3n+1)/2$. One can construct a graph by starting with a cycle and connecting pairs of vertices that are two apart. If it is odd length, the pairing leaves one out, and one can add an edge to that and any other vertex.

11. Give an example of a stable roommates instance with at least two stable pairings.

    Answer: Use the basic example of 2 men, women with two stable pairs and make the same sex preferences last.

12. Show how in any 2 by 2 instance of a stable marriage problem, the women can collaborate to produce a women optimal pairing. (Note: the men must follow the TMA algorithm, but the women could reject men who propose in an arbitrary fashion.) **Answer is short argument.**

    Answer:

    If both women have the same favorite man or both men have a single favorite women there is one stable pairing; the one that contains that favorite along with her/his partner.

    Otherwise the men and women all have different favorites. Thus, if the men propose on the first day, if both woman get their favorite, they accept, otherwise they reject and the men move on and then the women will get their favorites.

Yaay. Down with this Tradition!

An easier solution is that the women can all precompute their optimal partner in any matching instance and simple reject men until their optimal partner proposes.

3. Expression Proofs: 6/6/6

1. Prove: $n^2 \not\equiv 1 \pmod 7 \implies n \not\equiv 1 \pmod 7$
   Answer: Prove by contraposition.
2. Prove: $\forall n \in \mathbb{N}, n \geq 2 \implies (1-1/4)(1-1/9)\cdots(1-1/n^2) = \frac{n+1}{2n}$.
   Answer: Base Case: True for $n = 2$.
   Ind. Hyp: $(1-1/4)(1-1/9)\cdots(1-1/n^2) = \frac{n+1}{2n}$.
   Ind. Step.

$$\begin{aligned}(1-1/4)(1-1/9)\cdots(1-1/n^2)(1-1/(n+1)^2) &= \frac{n+1}{2n}(1-1/(n+1)^2)\\ &= \frac{n+1}{2n}(\frac{n+2n}{(n+1)^2}\\ &= \frac{n+1}{2n}(\frac{n(n+2)}{(n+1)^2}\\ &= \frac{n+2}{2(n+1)}\end{aligned}$$

3. Use induction to prove that $1+\frac{1}{2}+\cdots(\frac{1}{2})^n \leq 2$? (Hint: strengthen the statement.)
   Answer: Statement: $1+\frac{1}{2}+\cdots(\frac{1}{2})^n = 2-(\frac{1}{2})^n$
   Base Case: $n = 0$. Plug in and we get $1 = 2-(\frac{1}{2})^0$.

   Induction Step:

$$\begin{aligned}1+\cdots+(\frac{1}{2})^{n+1} &= 2-(\frac{1}{2})^n+(\frac{1}{2})^{n+1}\\ &= 2-((\frac{1}{2})^n-(\frac{1}{2})^{n+1})\\ &= 2-(\frac{1}{2})^{n+1}\end{aligned}$$

4. Graphs and modular: 5/5/5 points.

1. The integer a is a quadratic residue of n if $\gcd(a,n) = 1$ and $x^2 \equiv a \pmod n$ has a solution. Prove that if p is a prime number, $p \neq 2$, then there are $(p-1)/2$ quadratic residues of p among $\{1,\ldots,p-1\}$.
   Answer: Evaluate x^2 for each $x \in \{1,\ldots,p-1\}$. Notice that they appear in pairs because $x^2 = (-x)^2$. Now there are $(p-1)/2$ pairs of quadratic residues and one then argues that they are distinct as follows. Let $x^2 = y^2 \pmod p$ or $x^2-y^2 = (x-y)(x+y) = 0 \pmod p$ which implies that $x = y \pmod p$ and $x = -y \pmod p$.

2. Consider a directed graph where every pair of vertices u and v are connected by a single directed arc either from u to v or from v to u. Show that every vertex has a directed path of length at most two to **the vertex with maximum in-degree.** Note that this is quite similar to a homework problem but asks for a more specific answer. (Hint: Our solution doesn't require induction.)

   Answer: The total in-degree is the number of arcs which is $n(n-1)/2$ and thus the vertex v with maximum in-degree must have in-degree d at least $(n-1)/2$.

   Thus, these d vertices has a path of length 1 to v. The other vertices, of which there are $n-1-d$, have in-degree at most d and thus out-degree at least $n-1-d$, thus each must have an arc to one of the d vertices directly connected to v.

3. Consider a simple n-vertex graph that contains a path,$v_1,\ldots,v_n$, of length n and where the degrees of v_1 and v_n are at least $n/2$. Show the graph has a Hamiltonian cycle. (Recall a Hamiltonian cycle is one that visits each vertex exactly once.)

   Answer: If the ends are adjacent we simply add that edge to the path.

   Otherwise, consider the edges (v_i, v_{i+1}) with $i \neq 1$ and $i \neq n$, where (v_{i+1}, v_1) is an edge. That is their larger numbered endpoint is also adjacent to v_1.

   Notice there are at least $n/2-1$ such edges out of $n-3$ edges on their path.

   Similarly, consider the edges (v_i, v_{i+1}) where (v_i, v_n) is an edge with $i \leq n-1$. Here we are considering hitting the smaller numbered endpoint. Again there are at least $n/2-1$ such edges.

   Since $2(n/2-1) > n-3$, by the pigeonhole principle, there must there be an edge (v_i, v_{i+1}) where (v_i, v_n) and (v_1, v_{i+n}) are edges.

   Thus, the sequence $v_1,\ldots,v_i,v_n,v_{n-1},\ldots,v_{i+1}$ forms a cycle (since (v_i, v_n) and (v_{i+1}, v_1) are both edges in the graph as are (v_i, v_{i+1}).

5. Reverse Preference Stable Marriage: 10 points.

A witch bewitches all the men just before a run of TMA so they propose in reverse order of their preference lists (equivalent to the men giving reversed lists as inputs to TMA). The spell wears off after the algorithm is run and a pairing is obtained. (Note : ONLY MALES reverse their preference lists and after the spell wears off the men remember their original preferences)

1. Find the output pairings for the following two instances:

   Instance 1:
   Men
   A:1,2
   B:1,2

   Women
   1:B,A
   2:A,B

   Instance 2:
   Men
   A:1,2
   B:2,1

   Women
   1:A,B
   2:B,A

   Answers only

   Answer: $\{(A,2),(B,1)\}$, $\{(A,2),(B,1)\}$

2. Prove that the pairing is either unstable or all the women got their first choice male. (**Hint:** Consider how to translate the notion of stability for the TMA pairing in this case. There is no couple (w^*, m) such that w^* has m higher than her partner m^* and m has w^* higher on his reversed list than his partner w.)

Answer: As per the hint, in the output pairing, there is no couple (w^*, m) such that $(w^*$ has $m > m^*) \wedge (m$ has $w^* > w$ on reversed list). This is just because the SMA outputs a stable pairing wrt to its input preference lists. Now note that (m has $w^* > w$ on reversed list) is equivalent to (m has $w > w^*$ on original list). So we have, there is no couple (w^*, m) such that $(w^*$ has $m > m^*) \wedge (m$ has $w > w^*$ on original list). This means for any woman w^*,
Case 1: $w*$ prefers another man m to her partner m^*, then $\neg(m$ has $w > w^*$ on his original preference list) $\implies$ (m has $w^* > w$ on his original preference list) $\implies$ (w^*, m) is a rogue couple.
Case 2: w^* prefers no other man to her partner, i.e, she got her first choice.

6. Cycles, pairings, and circle of worry: 10 points.

1. Argue that any directed simple graph where every vertex has out-degree at least one has a directed cycle.

   Answer: Take a walk from a vertex, at each step one can leave since there is an outvertex unless you have visited before, in which case there is a cycle.

2. Consider the graph formed with vertices corresponding to the men and women in a stable marriage instance and edges according to two different stable pairings, S and S'. If a pair is in both pairings only include a single edge in G, which ensures it is a simple graph. Argue that there is a cycle of length strictly greater than 2.

   Answer: Each man and woman has degree two in the non-simple graph consisting of the union of S and S', thus all are involved in a cycle. For at least one, woman the edges go to different man. Any walk that starts at that woman must end at that woman, and must have length greater than two as the two men immediately before and after are not the same.

3. Define a man m as **feeling threatened by** another man m' with respect to a pairing S if (m, w) is in S and w likes m' better than m. We define the **male feeling threatened graph** for a stable pairing S as the **directed** graph whose vertices are men and with a directed arc for each pair (m, m') where m is feeling threatened by m'. Show that the male feeling threatened graph for the male optimal pairing has a cycle if there is more than one pairing. (Hint: using the previous parts may be helpful.)

   Answer: Consider the union of the male optimal, S, and woman optimal, S', pairings. There is a non-trivial cycle in this graph, due to the above. Consider the men in that cycle. For some pair (m, w) in that cycle, let $(m', w) \in S'$. Here w prefers m' to m since S' is female optimal, and m' is not m since the cycle is non-trivial. Thus, m has outdegree 1 in the associated male feeling threatened graph. By (a), the feeling threatened graph has a cycle.