1. Propositional Logic

$\boxed{\text{Def.}}$ A **proposition** is a sentence that declares a fact, that is either true or false, but not both.

E.g. Which of the following are propositions?

- Berkeley is a city in the US ✓ true prop.
- $1+1=3$ ✓, false prop
- Read this carefully. X, not declarative
- $x+y=z$ X, declares somethings that's neither true nor false

Use letters to denote **propositional variables** (P, Q, R), i.e. variables that represent propositions

The **truth value** of a proposition is true (T) or false (F) based on whether the proposition is a true proposition or not.

We can form new propositions from the old ones.

$\boxed{\text{Def.}}$ Let P, Q be prop(ositions).

- The **negation** of P, denoted $\neg P$ (or $\bar{P}$), is the prop "It is not the case that P".
- The **conjunction** of P and Q, denoted $P \wedge Q$, is the prop "P and Q".

- The **disjunction** of P and Q, denoted $P \vee Q$, is the prop "P or Q".

E.g. (we all love discrete math) $\wedge$ (we all love probability theory)
a true prop since both clauses are true. :)

Truth table for $\neg, \wedge, \vee$ you can think of them as functions.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	F

$\boxed{\text{Def.}}$ Let P, Q be prop.

hypothesis conclusion

- The **implication** $P \Rightarrow Q$ is the prop "if P, then Q".
- The **biconditional** $P \Leftrightarrow Q$ is the prop "P if and only if Q".

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

"(1+1=2) is necessary for (Yining to like math)"

"(Yining likes math) only if (1+1=3)"

"(The existence of unicorns) is sufficient for (1+1=2)"

"(Yining has a pet unicorn) if (unicorns exist)"

you can deduce anything from a false hypothesis!

1.1 Propositional Equivalence

Def. A compound prop. that is always true regardless of the truth values of the prop. variables that occur in it is called a <u>tautology</u>, i.e. last column in the truth table only has T.

E.g.

P	$P \vee \neg P$
T	T
F	T

so $P \vee \neg P$ is a tautology.

Def. Two compound prop. P and Q are <u>logically equivalent</u>, denoted $P \equiv Q$, if $P \Leftrightarrow Q$ is a tautology, i.e. the last columns match.

E.g.

P	Q	$\neg(P \wedge Q)$
T	T	F
T	F	T
F	T	T
F	F	T

P	Q	$\neg P \vee \neg Q$
T	T	F
T	F	T
F	T	T
F	F	T

<u>De Morgan's Laws</u> distribute and flip

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

E.g. $(P \Rightarrow Q) \equiv (\neg P \vee Q)$

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Rem. Which one is correct?

① $(P \Rightarrow Q) \equiv \underbrace{(Q \Rightarrow P)}_{}$ ← converse

✓ ② $(P \Rightarrow Q) \equiv \underbrace{(\neg Q \Rightarrow \neg P)}_{}$ ← contrapositive

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	T	T	F	F	T
T	F	F	T	F	T	F
F	T	T	F	T	F	T
F	F	T	T	T	T	T

E.g. I'm Yining $\Rightarrow$ I don't own an unicorn.

- converse: I don't own an unicorn $\Rightarrow$ I'm Yining.
- contrapositive: I own an unicorn. $\Rightarrow$ I'm not Yining.

2. First-Order Logic

Recall: Let $P(x)$ be the statement "$x > 3$".

$P(x)$ is not a prop, because $P(x)$ doesn't have a truth value.

$P(x)$ is called a propositional function.

Once we assign value to x, $P(x)$ becomes a proposition.

E.g. Let $P(x, y)$ be the propositional function "$x > y$".

Then $P(0,1)$ has truth value F.

$P(3,2)$ has truth value T.

Another way to create a proposition from propositional function involves quantifiers.

Def The domain of a propositional function $P(x)$ is the set of all possible values of x.

Def.

- We use $\forall x\, P(x)$ to denote "$P(x)$ for each value of x in the domain"
- We use $\exists x\, P(x)$ to denote "there exists one element x in the domain such that $P(x)$"

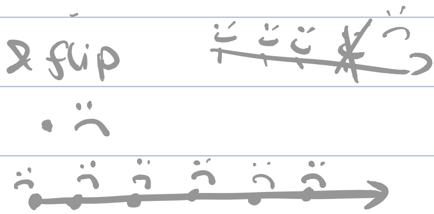
Rem. We often indicate the domain S of x using "$x \in S$"

E.g. What's the truth value of

the set of natural numbers $\{0,1,2,3,$

- $\forall x \in \mathbb{N},\ x \geq 0$ T
- $\exists x \in \mathbb{N},\ x \geq 0$ T

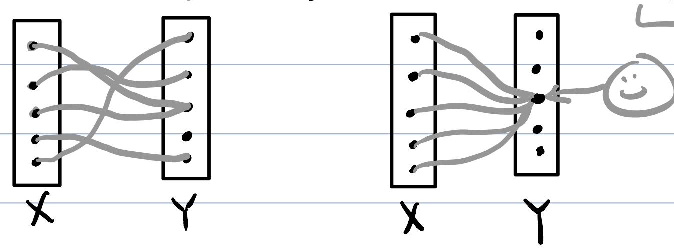
De Morgan's Laws distribute & flip



$$\neg(\forall x\ P(x)) \equiv \exists x, \neg P(x)$$

$$\neg(\exists x\ P(x)) \equiv \forall x, \neg P(x)$$

Rem: Is $\forall x \exists y\ P(x,y)$ the same as $\underbrace{\exists y\ \forall x\ P(x,y)}_{\text{stronger.}}$? No.



E.g. Assume we're in an universe with more than one person and more than one unicorn.

$\forall x \in \{\text{unicorns}\}, \exists y \in \{\text{human}\}$, y owns x
i.e. "each unicorn has a owner"

$\exists y \in \{\text{human}\}, \forall x \in \{\text{unicorns}\}$, y owns x
i.e. "there's someone who owns all unicorns".

<u>Fun</u>: One of us built JoinMe.

Amin: Not me. $\neg A$

Khalil: Not me. $\neg K$

Yining: Amin built it. A

Only one of us is telling the truth.

Who built JoinMe?

A : Amin built it , K: , Y:

We know $(A \wedge \neg K \wedge \neg Y) \vee (\neg A \wedge K \wedge \neg Y) \vee (\neg A \wedge \neg K \wedge Y)$ ①

and $(\neg A \wedge K \wedge \neg A) \vee (A \wedge \neg K \wedge \neg A) \vee (A \wedge K \wedge A)$

$\equiv (\neg A \wedge K) \vee (A \wedge K)$ ②

A	K	Y	①	②
⋮	⋮	⋮	⋮	⋮
F	T	F	T	T