Q: Yining talks to someone about math iff they don't talk to themselves about math.

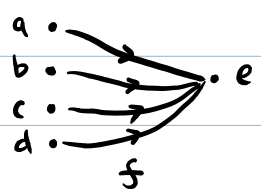
Does Yining talk to herself about math?

Q: $\{\phi\} \subseteq \mathbb{N}$?

- True
- False

Q: does f represent a well-defined function?

(a)



Yes!

(b) $f(x) = \frac{1}{x}$, $x \in \mathbb{R}$

No (not defined at 0).

1. Set

$\boxed{\text{Def}}$ A **set** is an unordered collection of objects.

E.g.
- $A = \{1,2,3\} = \{1,3,2\}$
- the empty set $\emptyset = \{\}$
- $\mathbb{N} = \{0,1,2,3,\dots\}$
- $\mathbb{Z} = \{\dots,-1,0,1,2,\dots\}$
- $\mathbb{Q} = \{p/q \mid p,q \in \mathbb{Z}, q \neq 0\}$
- $\mathbb{R}$, the set of real numbers

Notation: $a \in A$ means a is an element of the set A

$a \notin A$ means a is not an element of the set A

E.g.
- $0 \in \mathbb{N}$
- $0 \notin \emptyset$
- $\emptyset \notin \mathbb{N}$

Rem. (Russell's Paradox)

Let R be the set of all sets that are not members of themselves. Does R exist?

Let $R = \{x \mid x \notin x\}$.

$R \in R \Rightarrow R \notin R$; $R \notin R \Rightarrow R \in R$.

$\Rightarrow R \in R \Leftrightarrow R \notin R$; so R doesn't exist.

1.1 Comparing Sets

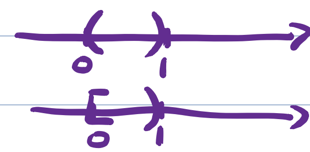
$\boxed{\text{Def.}}$ The set A is a **subset** of B, written $A \subseteq B$, if for each $a \in A$, we have $a \in B$.

$A \subset B$ means $A \subseteq B$, and $\underbrace{A \neq B}_{B \not\subseteq A}$.

$\forall a \in \emptyset \Rightarrow a \in \mathbb{N}$

E.g. • $\boxed{\emptyset \subset \mathbb{N}} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

• $(0,1) \subset [0,1) \subset [0,1] \subset \mathbb{R}$



$\boxed{\text{Def.}}$ Two sets A, B are **equal** if $A \subseteq B$ and $B \subseteq A$.

Recall: $P \Leftrightarrow Q$ means $P \Rightarrow Q$, $Q \Rightarrow P$.

1.2 New Sets from Old

e.g. S = {☺, ☹}. $\mathcal{P}$(S) = {∅, {☺}, {☹}, {☺, ☹}}

$\boxed{\text{Def.}}$ Given a set S, the **power set $\mathcal{P}(S)$ of S** is the set of all subsets of S.

An **ordered n-tuple** $(a_1, a_2, \dots, a_n)$ is the ordered collection has a_1 as its first element, a_2 as its second element, $\dots$, and a_n as its n^{th} element.

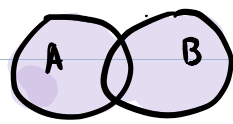
Def The Cartesian product $A_1 \times \cdots \times A_n$ is
$\{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}$.

E.g. the real plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

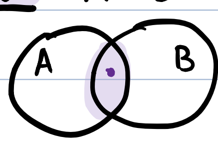
e.g. $A = \{1, 2\}$
$B = \{3, 4\}$
$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

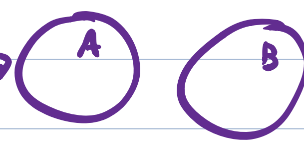
1.3 Set Operations

- union $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.



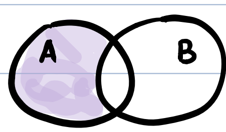
- intersection $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.



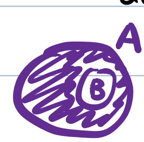


Rem: A and B are disjoint if $A \cap B = \phi$.

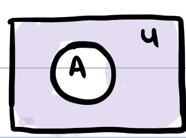
- difference $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.



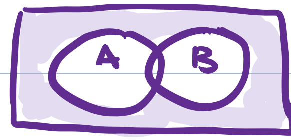
Rem. If $B \subset A$, we often use $A \setminus B$ to denote $A - B$.



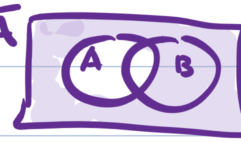
- complement $A \subset U$. $\bar{A} = \{x \in U \mid x \notin A\}$
" $U \setminus A$



$\overline{A \cup B}$



$\bar{A}$



E.g. Prove $\overline{A \cup B} = \bar{A} \cap \bar{B}$.

Pf: ① We will show $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Let $x \in \overline{A \cup B}$, that is $x \notin A \cup B$.

Thus, $x \notin A$ and $x \notin B$.

So $x \in \bar{A}$ and $x \in \bar{B}$.

Thus $x \in \bar{A} \cap \bar{B}$.

② We will show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Let $x \in \bar{A} \cap \bar{B}$, that is $x \in \bar{A}$ and $x \in \bar{B}$.

so $x \notin A$, and $x \notin B$.

Then $x \notin A \cup B$.

Thus, $x \in \overline{A \cup B}$. $\square$

2. Functions

$\boxed{\text{Def}}$ Let A, B be nonempty sets. A function f from A to B, written $f: A \to B$, is an assignment of exactly one element of B to each element of A.

E.g.:
- $A = \{\text{dog}\}$ $B = \{\text{animal}, \text{furniture}\}$.
  $f(\text{dog}) = \text{animal}$
- $f: \mathbb{N} \to \mathbb{N}$ such that $f(x) = x + 1$
- a computer program

$\boxed{\text{Def.}}$ If $f: A \to B$, we say A is the **domain** of f, and B is the **codomain** of f.

$\boxed{\text{Def}}$ Let $g: A \to B$ and $f: B \to C$ be functions. The **composition of f and g**, denoted $f \circ g$, is a function from A to C defined by $(f \circ g)(a) = f(g(a))$.

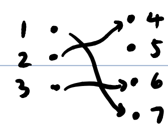
$$
\begin{tikzcd}
\overset{a}{A} \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \overset{g(a)}{B} \arrow[r, "f"] & \overset{f(g(a))}{C}
\end{tikzcd}
$$

2.1 Injection and Surjection

$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.

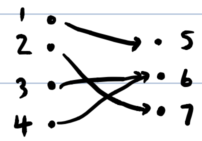
$\boxed{\text{Def}}$ A function $f: A \to B$ is **injective** or **one-to-one** if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, for all $a_1, a_2 \in A$.

E.g.

-  injection.
- $f: \mathbb{N} \to \mathbb{Z}$ such that $f(x) = |x|$ (x) injection.
- $f: \mathbb{Z} \to \mathbb{N}$ such that $f(x) = |x|$ not injection
- $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = |x|$ not injection

$\boxed{\text{Def}}$ A function $f: A \to B$ is **surjective** or **onto** if for any $b \in B$, there is an element $a \in A$ s.t. $f(a) = b$.

E.g.

-  Surjection
- $f: \mathbb{N} \to \mathbb{Z}$ such that $f(x) = |x|$ not Surjection.
- $f: \mathbb{Z} \to \mathbb{N}$ such that $f(x) = |x|$ Surjection.
- $f: \mathbb{Z} \to \mathbb{Z}$ such that $f(x) = |x|$ not a Surjection.

Rem. To show f is injective, show $\forall a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

To show f is surjective, show $\forall b \in B$, find $a \in A$, s.t. $f(a) = b$.

analogy: A labelled balls, B labelled bins.

f: person.

2.2 Bijections

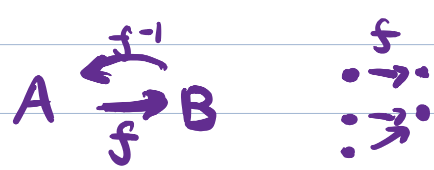
$\boxed{\text{Def.}}$ $f: A \to B$ is a bijection or one-to-one correspondence if f is both injective and surjective.

E.g.

- identity function $i_A: A \to A$ bijection.
- $f: \mathbb{R} \to \mathbb{R}^+$ where $f(x) = e^x$
- $f: \mathbb{R}^+ \to \mathbb{R}$ where $f(x) = \log x$

$\}$ bijection.

$\boxed{\text{Def}}$ Let $f: A \to B$ be a bijection. The inverse function of f, denoted $f^{-1}: B \to A$, is defined by $f^{-1}(b) = a$ when $f(a) = b$.



Rem: Let $f: A \to B$ be a bijection.

- $f^{-1} \circ f = id_A$; $f \circ f^{-1} = id_B$
- $(f^{-1})^{-1} = f$

$$\begin{tikzcd} A \arrow[r, "f"'] & B \arrow[l, bend right, "f^{-1}"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend right, "f"'] & B \arrow[l, bend right, "f^{-1}"'] \end{tikzcd}$$

E.g.

- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x$. $f^{-1}(y) = y$..
- $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x+1$. $f^{-1}(y) = y - 1$.
- $f: \mathbb{R} \to \mathbb{R}^{+}$ with $f(x) = e^{x}$. $f^{-1}(y) = \log y$

nonnegative real.

edit: positive reals!!

$f: \mathbb{R} \to \{\text{nonnegative reals}\}$ is not a bijection.

Do you see why?