Q : There're more real numbers in $[0, \frac{1}{10000}]$ than all rational numbers.

- True
- False

1. Cardinality of sets

Def Let S be a set. If there are exactly $n \in \mathbb{N}$ distinct elements in S, we say S is a finite set with cardinality n.

Notation. $|S|$ denotes cardinality of S.

E.g.
- $A = \{n \in \mathbb{N} \mid n < 5\}$. $|A| = 5$. $A = \{0,1,2,3,4\}$
- $|\phi| = 0$
- $|\{0\}| = 1$

Def A set is said to be **infinite** if it is not finite.

Recall : $f : A \to B$ A : labelled balls, B : labelled bins.

Def Two sets A and B have the **same cardinality**, written $|A| = |B|$, if there is a bijection from A to B.

E.g. Let S be the set of even integers. Prove that $|S| = |\mathbb{Z}|$.

Pf:
$S = \{\dots, -4, -2, 0, 2, 4, \dots\}$
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Consider $f : \mathbb{Z} \to S$ such that $f(n) = 2n$.

① [To show f is injective] $f(a) = f(b) \Rightarrow a = b$.

Suppose $f(a) = f(b)$ for some $a, b \in \mathbb{Z}$.

Thus, $2a = 2b. \Rightarrow a = b.$

Thus, f is injective.

② [To show f is surjective] $\mathbb{Z} \xrightarrow{f} S$

Let $s \in S$. $? \longmapsto s$

Then $\frac{1}{2}s \in \mathbb{Z}$. Furthermore, $f(\frac{1}{2}s) = 2 \cdot \frac{1}{2} \cdot s = s$.

Thus, f is surjective. $\square$

Def A set that is finite or has the same cardinality as $\mathbb{N}$ is called countable.

Rem. An infinite set S is countable if we can list elements in S in a sequence $a_0, a_1, a_2, \ldots$ because $f : \mathbb{N} \to S$ given by $f(n) = a_n$ is a bijection.

E.g. • $\mathbb{Z}$ is countable.

$0, 1, -1, 2, -2, 3, -3, \ldots$

• The set of finite length bit strings is countable.

$0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots$

Thm (Schröder - Bernstein) If there exist injections $f : A \to B$ ($|A| \leq |B|$.) and $g : B \to A$ ($|B| \leq |A|$.) between sets A and B, then there exists a bijection $h : A \to B$.

$\boxed{\text{Cor.}}$ $\mathbb{Q}^+$ is countable.

Pf: Obviously, there's an injection from $\mathbb{N}$ $\{0,1,2,\dots\}$ to $\mathbb{Q}^+$.

We need to find an injection from $\mathbb{Q}^+$ to $\mathbb{N}$.

Recall that $\mathbb{Q}^+ = \{ p/q \mid p, q \in \mathbb{Z}^+ \}$.

$$
\begin{array}{ccccc}
a_0\ \frac{1}{1} & \overset{a_1}{\frac{2}{1}} & \frac{3}{1} & a_5\ \frac{4}{1} & \cdots \\
a_2\ \frac{1}{2} & \frac{2}{2}\ a_4 & \frac{3}{2} & \frac{4}{2} & \cdots \\
a_3\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \cdots
\end{array}
$$

so $q \longmapsto \min\{ n : a_n = q \}$ is an injection from $\mathbb{Q}^+$ to $\mathbb{N}$.

$\Rightarrow |\mathbb{Q}^+| = |\mathbb{N}|$. $\square$.

Rem. It follows that $\mathbb{Q}$ is countable as well.

1.1 Cantor diagonalization argument

$\boxed{\text{Thm}}$ $\mathbb{R}$ is uncountable.

Pf: Assume $\mathbb{R}$ is countable.

Since $[0,1] \subset \mathbb{R}$, $\overbrace{,}^{\text{pf. exercise.}}$ $[0,1]$ is countable.

List elements in $[0,1]$ as $r_0, r_1, r_2, \ldots$.

Let the decimal representation of them as.

$r_0 = 0.d_{00}d_{01}d_{02}\cdots$ $\qquad$ $r_0 = 0.00000\cdots$

$r_1 = 0.d_{10}d_{11}d_{12}\cdots$ $\qquad$ $r_1 = 0.1415926\cdots$

$r_2 = 0.d_{20}d_{21}d_{22}\cdots$ $\qquad$ $r_2 = 0.3261\cdots$

$\vdots$ $\qquad$ $\vdots$

Form a real number with decimal expansion $\quad r = 0.100\cdots$

$r = 0.d_0 d_1 d_2 \cdots$

Such that $d_i = \begin{cases} 1 & \text{if } d_{ii} = 0 \\ 0 & \text{if } d_{ii} \neq 0 \end{cases}$ $\quad$ ith digit of r_i

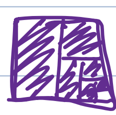
Then r differs at the ith digit with r_i, so $\forall i$, $r \neq r_i$.

$\Rightarrow r$ is a real number not on our list.

Hence, $[0,1]$ is not countable, so $\mathbb{R}$ is not countable. $\square$.

Rem. Similarly, the set of infinite length bit strings is uncountable.

Rem. Be careful with uncountable sets!

$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 1.$  However $\sum_{r\in\mathbb{R}} x_r = \infty$. $\quad x_r > 0$

$$\underbrace{\overset{\text{finite}}{\emptyset} \subset \overbrace{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}}^{\text{infinite}}}_{\text{countable.}} \subset \underbrace{\mathbb{R}}_{\text{uncountable}}$$

2. Uncomputable Functions

$\boxed{\text{Def}}$ A function is **computable** if there is a computer program in some programming language that finds the value of this function.

$\boxed{\text{Thm}}$ There are uncomputable functions.

Pf: Claim: There're uncountably many functions from $\mathbb{N}$ to $\mathbb{N}$.

Pf: Suppose there're countably many functions from $\mathbb{N}$ to $\mathbb{N}$.

	0	1	2	...
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	...
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	...
f_2	$f_2(0)$	$f_2(1)$	$f_2(2)$	...
$\vdots$				

f $\quad f_0(0)+1 \quad f_1(1)+1 \quad f_2(2)+1.$

$f: \mathbb{N} \to \mathbb{N}$ not on our list.

$\Rightarrow$ . . . . . . conclude. $\square$.

A computer program is a bit string with finite length.

$\Rightarrow$ $\{$computer programs$\}$ is countable.

$\Rightarrow$ there're are uncomputable functions. $\square$.

	P_0	P_1	P_2	$\cdots$	Turing
P_0	L	H	H	$\cdots$	
P_1	H	H	L		
P_2	L	L	L		
$\vdots$ Turing	$\vdots$				

2.1 uncomputable function: an example

Define $\text{TestHalt}(\overset{\text{program}}{P}, \overset{\text{input}}{x}) = \begin{cases} \text{"yes"} & \text{if program } P \text{ halts on input } x \\ \text{"no"} & \text{if program } P \text{ loops on input } x \end{cases}$

$\boxed{\text{Thm}}$ TestHalt is uncomputable.

Pf: Assume TestHalt is computable.

Define

$\text{Turing}(P) = \begin{cases} \text{loop forever.} & \text{if TestHalt}(P,P) == \text{"yes"} \\ \text{halt} & \text{if TestHalt}(P,P) == \text{"no"} \end{cases}$

P(P) halts.

What is Turing(Turing) ?

If Turing(Turing) halts,

$\Rightarrow$ TestHalt(Turing, Turing) == "no".

$\Rightarrow$ Turing(Turing) loops forever

If Turing(Turing) loops forever,

$\Rightarrow$ ...

$\Rightarrow$ ...

$\square$.

Rem. A common strategy to show a program P is uncomputable is using P to implement testhalt.

i.e. "reducing TestHalt to P".

P computable $\Rightarrow$ TestHalt computable.

$\boxed{\text{Prop}}$ A is countable. Given $B \subseteq A$, then B is countable.

Pf: The statement obviously holds if A or B is finite.

So assume A, B are infinite.

A is countable $\Rightarrow \exists$ bijection $f : A \to \mathbb{N}$

Restrict f on $B \subseteq A$ to get $f : B \to \mathbb{N}$, an injection.

Then $f : B \to \underbrace{f(B)}_{\subseteq \mathbb{N}}$ is a bijection.

Claim: An infinite subset N of $\mathbb{N}$ is countable.

Pf (of the claim): Define $g : \mathbb{N} \to N$ recursively by

$$\begin{cases} g(0) = \min N. \\ g(n+1) = \min \{ n \in N \mid n > g(n) \}. \end{cases}$$

Then by construction, $g(0)$ is a bijection.

Since $f(B)$ is an infinite subset of $\mathbb{N}$, by the claim, $f(B)$ is countable, i.e. there exists a bijection $g : f(B) \to \mathbb{N}$.

Thus, $g \circ f : B \to \mathbb{N}$ is a bijection, i.e. B is countable.

$$B \xrightarrow{f} f(B) \xrightarrow{g} \mathbb{N}$$