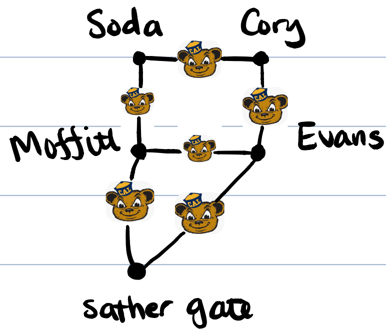
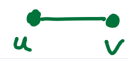
This is Oski



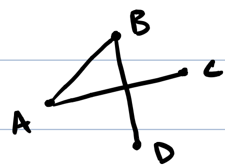
Q : Is it possible to start & end at sather gate, such that you visit each oski exactly once?

- possible
- impossible

1. Graphs



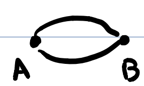
$\boxed{\text{Def}}$ An (undirected simple) **graph** $G = (V, E)$ is defined by a set of **vertices** V and a set of **edges** E, where elements in E are of form $\{u, v\}$ where $u, v \in V$, $u \neq v$.

E.g. ① 

$\Longleftrightarrow$ $V = \{A, B, C, D\}$

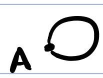
$E = \{\{A, B\}, \{B, D\}, \{A, C\}\}$.

no multiple edges

② 

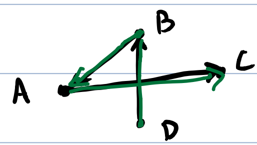
not a simple graph. b/c. $E = \{\{A, B\}, \{A, B\}\}$??? not a set

no self-loop.

③ 

not a simple graph. b/c $\{A, A\}$ not a set.

Rem. To model a directed graph $G = (V, E)$, we can define $E \subseteq V \times V$.



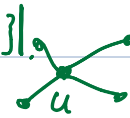
$V = \{A, B, C, D\}$.

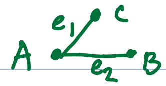
$E = \{(B, A), (A, C), (D, B)\}$.

$\boxed{\text{Def}}$ Given an edge $e = \{u, v\}$, we say 

- e is **incident** on vertices u and v;
- u and v are **neighbors** or **adjacent**

The degree of a vertex u is $|\{v \in V : \{u, v\} \in E\}|$.





$m = 2$, (B, e_2) (A, e_2) (A, e_1) (C, e_1)

Thm (The handshaking theorem) Let $G = (V, E)$ be a graph with m edges. Then $2m = \sum_{v \in V} \deg(v)$.

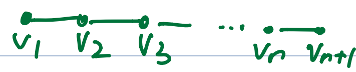
Pf: Let N be the number of pairs (v, e) such that v is an endpoint of e.

Since each v belongs to $\deg(v)$ pairs, $\sum_{v \in V} \deg(v) = N$.

On the other hand, each edge belongs to 2 pairs, so $N = 2m$.

Hence, $2m = \sum_{v \in V} \deg(v)$. $\square$

1.1 Eulerian Tours

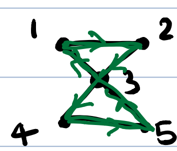


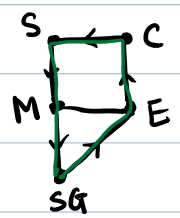
Def. A **walk** is a sequence of edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_n, v_{n+1}\}$.

A **tour** is a walk that has no repeated edges, starts and ends at the same vertex.

A **Eulerian tour** is a tour that visit each edge exactly once.

Rem. A walk can be specified by a sequence of vertices in the order of visit.

E.g. ① An Eulerian tour in  is 1, 2, 3, 4, 5, 3, 1.

② 

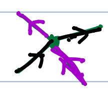
SG, E, C, S, M, SG is not an Eulerian tour because $\{M, E\}$ is not visited.

$\boxed{\text{Def}}$ A graph is **connected** if there exists a path between any distinct $u, v \in V$.

$\boxed{\text{Thm.}}$ A connected graph G has an Eulerian tour iff every vertex has even degree.

Pf: ① ("$\Rightarrow$") Assume G has an Eulerian tour starting at v_0.

For all $v \in V$, pair up the two edges each time we enter and exit.



For v_0, additionally pair up the starting edge, and the ending edge.

Eulerian tour visits all edges exactly once,

$\Rightarrow \forall v \in V$, incidicent edges are paired

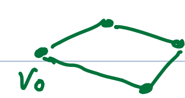
$\Rightarrow \forall v \in V$, $\deg(v)$ is even.

② ("$\Leftarrow$") Assume every vertex in G has even degree.

Goal: Find an Eulerian tour.

Step 1: Pick an arbitrary $v_0 \in V$ to start.

Keep following unvisited edges until stuck.



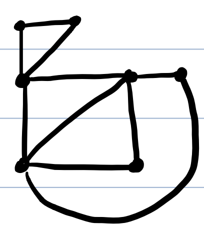
All degrees even $\Rightarrow$ stuck at V_0.

Step 2: Remove this tour.

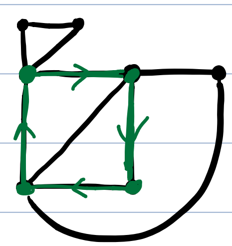
Recurse on connected components.

Step 3: Splice the recursive tours into the main one to get a Eulerian tour.

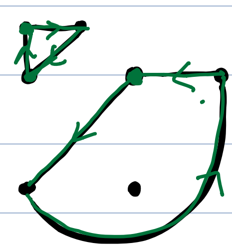
E.g. Use the algorithm above to find an Eulerian tour in the following graph.



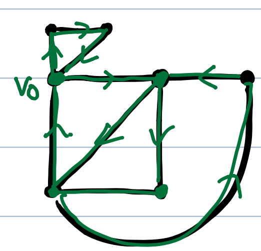
Step 1:



Step 2:



Step 3:

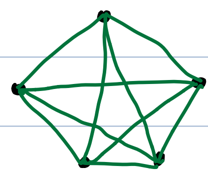


2. Special Graphs

2.1 Complete Graphs

a **complete graph** with n vertices, denoted K_n, is a graph that contains every possible edge.

E.g. K_5

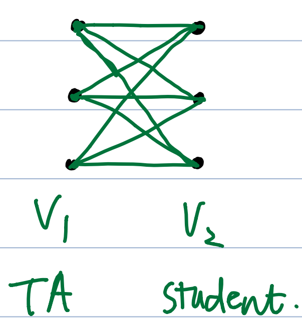


2.2 Bipartite Graphs

bipartite graph partitions vertices V into two disjoint sets V_1 and V_2 such that $E \subset \{\{u,v\} : u \in V_1, v \in V_2\}$.

a **complete bipartite graph** has $E = \{\{u,v\} : u \in V_1, v \in V_2\}$, denoted $K_{|V_1|,|V_2|}$

E.g. $K_{3,3}$

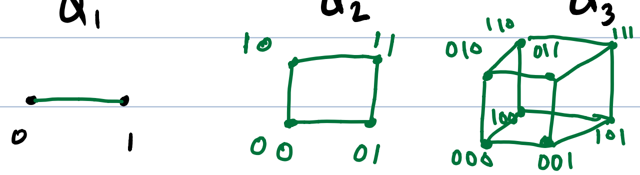


2.3 Hypercubes

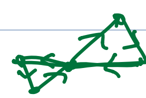
An **n-dim hypercube**, denoted Q_n, has a vertex for each length-n bit string, and an edge between a pair of vertices iff they differ in one bit.

<u>Rem</u>. Hypercubes can be constructed recursively.
To build Q_{n+1} from Q_n,
- make two copies of Q_n,
- prefacing 0 for one copy and 1 for the other.
- add edges between copies of corresponding vertices.

<u>E.g.</u> Q_1 Q_2 Q_3



2.4 Trees



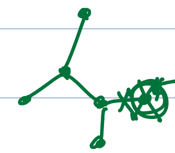
$\boxed{\text{Def}}$ A cycle is a tour, s.t. the only repeated vertex is the start and end vertex.

$\boxed{\text{Def}}$ A **tree** is a connected, acyclic graph.
A **leaf** is a vertex of degree 1.

Rem. Try to prove leaf lemmas :

"every tree has at least one leaf" and

"a tree minus a leaf remains a tree".



They allow us to do induction on trees !!!

Thm T is a tree (connected, no cycle.)

$\Leftrightarrow$ $T = (V, E)$ is connected and has $|V| - 1$ edges

Pf: ① ("$\Rightarrow$") We'll do induction on $n = |V|$, i.e.,

$P(n)$: tree T has n vertices $\Rightarrow$ T has $n-1$ edges.

Base case : $n = 1$. • $n-1 = 0$. ✓

$P(n-1) \Rightarrow P(n)$. Inductive Step : Suppose T has n vertices.

By leaf lemmas, we can remove a leaf & its incident edge to get a tree T' with $n-1$ vertices.

By IH, T' has $(n-1)-1 = n-2$ edges.

$\Rightarrow$ T has $n-2+1 = n-1$ edges.

② ("$\Leftarrow$") We'll do induction on $n = |V|$.

$P(n)$: T is connected, has $n-1$ edges $\Rightarrow$ T is a tree (connected, no cycle.)

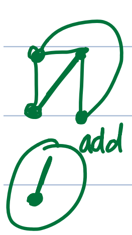
Base case: $n = 1$. • ✓

$P(n-1) \Rightarrow P(n)$. Inductive Step: Suppose T connected, has $n-1$ edges. $|V|$.

By handshaking theorem, total degree $= 2(n-1) = 2n-2$

$\Rightarrow \exists v \in V, \deg(v) = 1$. Remove a vertex v of deg 1 and its incident edge.

total deg $= \sum_{v\in V} \deg v = 2n-2$
no vertex of deg 1 ×
$\Rightarrow \forall v \in V, \deg(v) \geq 2$.
$\Rightarrow \sum_{v\in V} \deg(v) \geq 2n$



add.

to get T' that has $n-1$ vertices, and $n-2$ edges.

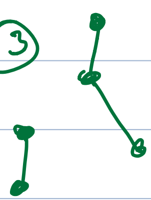
By IH, T' is connected, no cycle.

Now, adding back v and its edge, we still get a connected graph, and creates no cycles. $\Rightarrow$ T is a tree. □

Def A cycle is a tour where the only repeated vertices are the start and end vertices.

Thm The following statements are all equivalent:

- T is connected and contains no cycle.
- T is connected and has $|V|-1$ edges.
- T is connected, and removing any edge disconnects T.
- T has no cycle, and adding any single edge creates a cycle.

3 

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