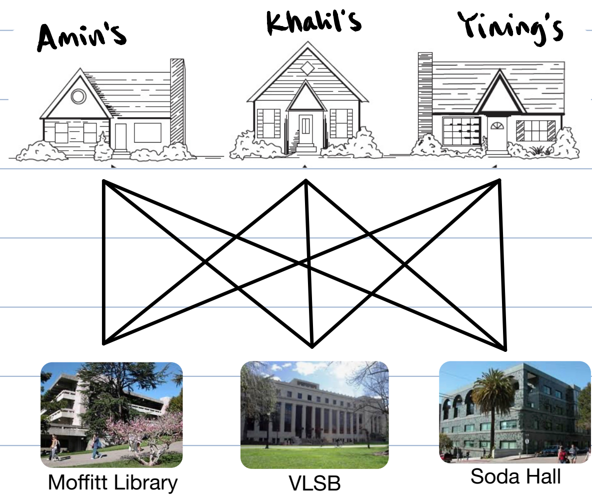
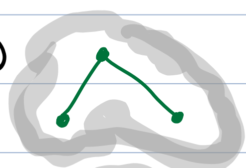
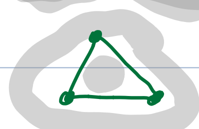
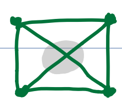
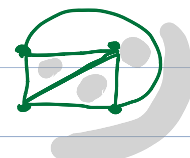


Q: Is it possible to design the streets without crossing?

1. Planar Graphs

Def A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

		v	e	f	
E.g. ①		3	$-\ 2$	$+\ 1$	$= 2$
②		3	$-\ 3$	$+\ 2$	$= 2$
③	 $\equiv$ 	4	$-\ 6$	$+\ 4$	$= 2$

Thm (Euler's Formula) G is a connected planar graph.

Let v = # vertices, e = # edges,

f = # faces in a planar representation of G.

Then $v - e + f = 2$.

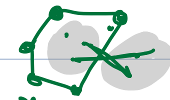
Pf: We'll do induction on e

Base case: $e = 0$ • $1 - 0 + 1 = 2$. ✓

Inductive Step: Consider G connected, has e edges.

- If G is tree, then $V = e + 1$, $f = 1$.
  Since $(e+1) - e + 1 = 2$, the formula holds.

- If G is not a tree, there's a cycle.



Take any cycle and remove an edge from it.
The resulting graph has v vertices, $e-1$ edges, and $f-1$ faces.
By IH, $v-(e-1)+(f-1) = v-e+f = 2$

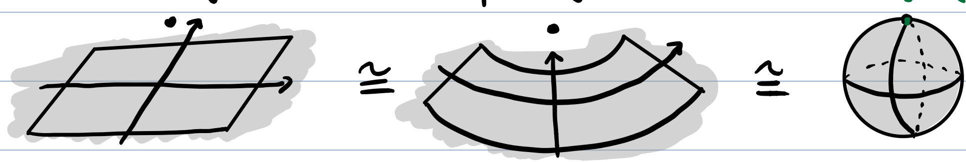
$\square$.

Rem. "$(-1)^0 \cdot (\#\ \overset{v}{0}\text{-dim things}) + (-1)^1 \cdot (\#\ \overset{e}{1}\text{-dim things}) + (-1)^2 \cdot (\#\ \overset{f}{2}\text{-dim things}) = 2$".

not in scope! but fun.

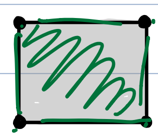
Does 2 feel unnatural?

It comes from the "shape" of $\mathbb{R}^2 \cup \{\infty\}$ surface of a sphere.



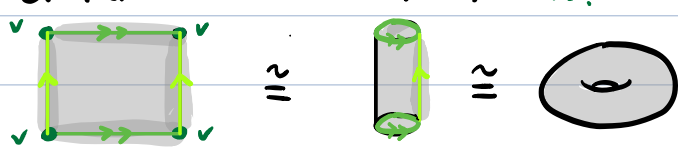
Similarly, different integers can be assigned to different surfaces.

For example, take a square,



$v = 4,\ e = 4,\ f = 1 \Rightarrow v-e+f = 4-4+1 = 1.$ ☺

Or take a "donut surface", torus.



$v = 1,\ e = 2,\ f = 1 \Rightarrow v-e+f = 1-2+1 = 0$ ☺

And!! This notion generalizes to higher dimensional objects:

$\chi = b_0 - b_1 + b_2 - b_3 + \cdots$.

↖ Euler characteristic.

1.1 Sparsity

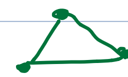
$\boxed{v-e+f=2}$

Cor. For a connected planar graph with $v \geq 3$, we have $e \leq 3v-6$.

Pf: Define the degree of a face to be the # edges on the boundary of the face.

e.g.

$\deg(F_1)=3$ $\deg(F_2)=5$



Then $\sum_{i=1}^{f} \deg(F_i) = 2e$

Since $\deg(F_i) \geq 3$ for any i, $2e \geq 3f$.

$\Rightarrow f \leq \frac{2}{3}e$. ①

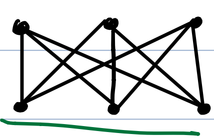
By planarity, $v-e+f=2$. $\Rightarrow f = 2+e-v$.

$\overset{①}{\Rightarrow} 2+e-v \leq \frac{2}{3}e$

$\Rightarrow e \leq 3v-6$. □.

Rem. This corollary says planar graph has "few" edges.

E.g. ① Is $K_{3,3}$ planar? planar $\Rightarrow e \leq 3v-6$.

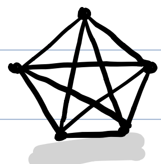


$e=9$. $3v-6 = 3\times 6-6 = 12$.

$e \leq 3v-6$.

We don't know...

② Is K_5 planar?



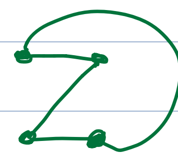
$e = 10$. $3v-6 = 3\times5-6 = 9$.

$e \not\leq 3v-6$.

nonplanar!

$\boxed{\text{Cor}}$ For a connected ~~non~~planar bipartite graph with $v \geq 3$, we have $e \leq 2v-4$

<u>Pf</u>: Similarly, $\boxed{\sum_{i=1}^{f} \deg(F_i) = 2e}$.



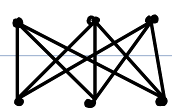
Now, $\deg(F_i) \geq 4$ for all i.

$\Rightarrow$. $2e \geq 4f = 4(2+e-v) = 8+4e-4v$

$\Rightarrow$ $4v-8 \geq 2e$.

$\Rightarrow$ $e \leq 2v-4$. $\square$.

<u>E.g.</u> Is $K_{3,3}$ nonplanar?



$e=9$. $2v-4 = 2\times6-4 = 8$.

$e \not\leq 2v-4$.

$K_{3,3}$ is nonplanar!!

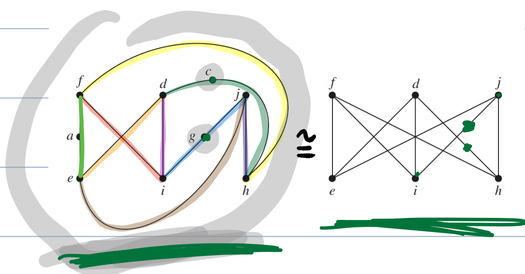
1.2 Kuratowski's Theorem

Def An operation on G by removing an edge $\{u,v\}$ and adding a new vertex w together with edges $\{u,w\}$, $\{v,w\}$ is an **elementary subdivision**.

Rem. If G is planar, after performing an elementary subdivision on G, G remains planar.

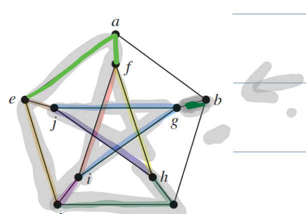
Def G_1 and G_2 are **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.

E.g.



Thm A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

E.g.



The Peterson Graph is nonplanar.

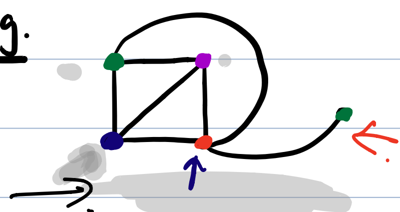
2. Graph Coloring

e.g. vertices : students
edges : friends. coloring : breakout rooms

Def A coloring of a graph G is the assignment of a color to each vertex such that no two adjacent vertices are assigned the same color.

The chromatic number $\chi(G)$ is the least number of colors needed for a coloring of this graph.

E.g.



$\chi(G) = 4$.

Prop. G is a planar graph $\Rightarrow \chi(G) \le 6$ // $\sum_{v \in V} \deg(v)$.

Pf: Since $e \le 3v - 6$, total degree (vertices) $2e \le 6v - 12$.

$\Rightarrow$ average degree $\frac{6v-12}{v} = 6 - \frac{12}{v} < 6$.

$\Rightarrow \exists v \in V$ s.t. $\deg(v) \le 5$.

We'll now do induction on $|V|$.

Base case: $|V| = 1$. • $\chi(G) = 1$. ✓.

Inductive Step: Remove a vertex v with degree ≤ 5.

By IH, the resulting subgraph G' has $\chi(G') \le 6$.

Color G' using ≤ 6 colors $\{c_1, c_2, c_3, c_4, c_5, c_6\}$.

Now color v.

Since $\deg(v) \le 5$, there's an available color among $c_1, \dots, c_6$.

$\square$.

$\boxed{\text{Thm.}}$ (5-color Theorem) G is a planar graph $\Rightarrow \chi(G) \le 5$.

Pf: Again, we'll do induction on $|V|$.

Base case: $|V| = 1$. $\chi(G) = 1$.

Inductive Step: Remove a vertex v with $\deg \le 5$.

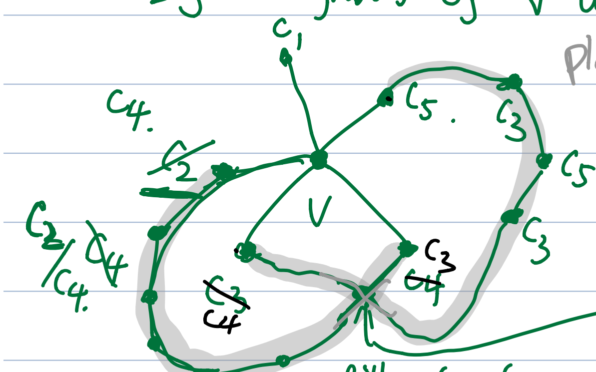
By IH, color the resulting G' using $\{c_1, \dots, c_5\}$.

Now need to color v.

If neighbors of v don't use up all five colors, color v using a remaining color.

If neighbors of v use up $\{c_1, \dots, c_5\}$,

planar representation.



you can alway flip c_2/c_4 or c_3/c_5 to free a color.

either c_2 or c_4
either c_3 or c_5.

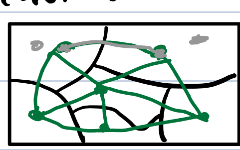
So no valid coloring for this vertex. $\square$

$\boxed{\text{Thm}}$ (4-Color Theorem) G is a planar graph $\Rightarrow \chi(G) \le 4$.

Rem. 5-color theorem was proven in 1800s.

4-color theorem was proven in 1976.

Rem. 4 color theorem tells us 4 colors are enough to color maps.



map $\longrightarrow$ graph.