Counting 1

CS 70, July, 14, 2020

Would like to answer questions like:

How many outcomes ossible for k coin tosses?
How many Poker hands?
How many handshakes for n People?
How many diagonals in a convex Polygon?
How many 10 digits number?
How many 10 digits numbers with out repetition?

Today's Topics:

1) First Rule of counting
2) Second Rule of counting
3) Sum Rule
4) Combinatorial Proofs

How many 3-bit string? or ...
How many different sequences of three bits from $\{0,1\}$?

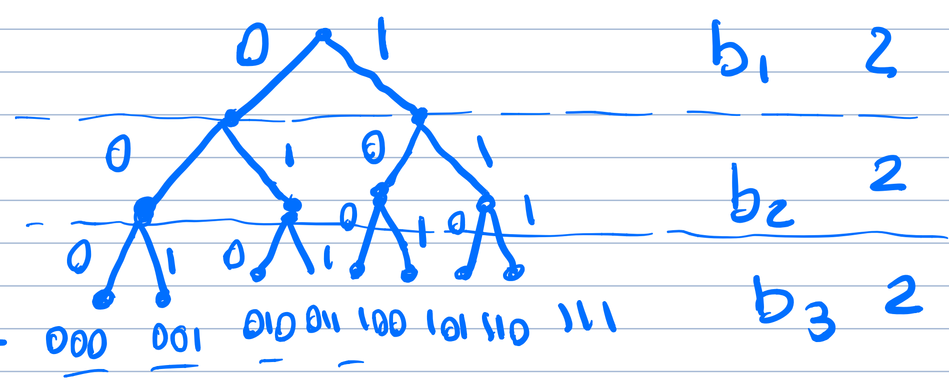
It's a sequence of numbers: $\underline{b_1}$ $\underline{b_2}$ $\underline{b_3}$

Use a binary tree

$\Rightarrow 2 \times 2 \times 2 = \underline{8}$

8 3-bit string

8 leaves.



1) First Rule of Counting: Product Rule

For an object that can be made by a sequence of choices, such that there are n_1 choices, then n_2 choices ..., then n_k steps

The number of objects to make is: $n_1 \times n_2 \times \cdots \times n_k$

Some Examples:

How many outcomes possible for k coin tosses?

outcomes: heads or tails

so 2 ways for the first choice, 2 way for second ... $\Rightarrow \underbrace{2 \times 2 \times \cdots \times 2}_{k} = 2^k$

How many 10 digits number? k digits

$\underbrace{1 \quad 2 \quad 3 \quad \cdots \quad k}_{k}$

First choice 10
second " 10
$\vdots$
k^{th} choice 10

$\Rightarrow 10 \times 10 \times \cdots \times 10 = 10^k$.

How many $\underline{n}$ digit base $\underline{m}$ numbers?

First digit m
Second " m
$\vdots$
n^{th} digit m

$$\Rightarrow \underbrace{m \times m \times \cdots \times m}_{n} = m^n.$$

What if we assume $\underline{08}$ is not a $\underline{\text{two}}$ digit number?

n digit base m number?

First digit $m-1$
Second digit m
$\vdots$
n^{th} digit m

$$= (m-1) \times \underbrace{m \times \cdots \times m}_{n-1}$$

$$= (m-1)\, m^{n-1}$$

Functions: How many functions $\underline{f}$ mapping $\underline{S}$ to $\underline{T}$.

domain, $\underline{S}$	codomain: T
s_1	t_1
s_2	t_2
$\vdots$	$\vdots$
$s_{\lvert S\rvert}$	$t_{\lvert T\rvert}$

$f(s_1)$ $\lvert T\rvert$ ways
$f(s_2)$ $\lvert T\rvert$ ways
$\vdots$ $\vdots$
$f(s_{\lvert S\rvert})$ $\lvert T\rvert$ way

$$\Rightarrow \underline{\lvert T\rvert^{\lvert S\rvert}}$$

Polynomials: How many Polynomials of degree $d = K$ modulo P?

$$P(X) = \underbrace{a_K}_{} X^K + \underbrace{a_{K-1}}_{} X^{K-1} + \cdots + \underbrace{a_0}_{} X_0$$

$a_K \to P$
$a_{K-1} \to P$
$\vdots \quad \vdots$
$a_0 \to P$

$$\Rightarrow \underbrace{P \times P \times \cdots \times P}_{K+1} = \underbrace{P^{K+1}}_{}$$

How many non-zero Polynomials?

$P^{K+1} - 1.$

Permutations: How many 10 digits numbers without repeating a digit?

first digit 10
second digit 9
$\vdots$
tenth digit 1

$$\Rightarrow \underbrace{10 \times 9 \times 8 \times \cdots \times 1}_{} = 10!$$

factorial

How many different samples of K from n numbers without replacement.

first choice n
Second choice $n-1$
$\vdots$
K^{th} choice $(n-K+1)$
$\hookrightarrow n-(K-1)$

$$\Rightarrow n \times (n-1) \times \cdots \times (n-K+1)$$

$$= \frac{n \times (n-1) \times \cdots \times (n-K+1) \times (n-K) \times \cdots \times 1}{(n-K) \times (n-K-1) \times \cdots \times 1}$$

$$= \frac{n!}{(n-K)!}$$

How many ordering of n objects are there?

- Permutations of n objects

First choice n
second " $(n-1)$
$\vdots$
n^{th} choice 1

$\Rightarrow n \times (n-1) \times \dots \times 1 = n!$

How many one-to-one function from S to S

$|S|$

$f(s_1) \rightarrow |S|$
$f(s_2) \rightarrow |S|-1$
$\vdots$
$f(s_{|S|}) \rightarrow 1$

$= |S| \times (|S|-1) \times \dots \times 1$

$= |S|!$

How many poker hands? (5 cards from 52)

first 52
second 51
⋮
fifth 48

$$\frac{52 \times \cdots \times 48 \times 47 \times \cdots \times 1}{47 \times \cdots \times 1} = \frac{52!}{47!}$$

Are hands {A, K, Q, J, 10} and {K, J, Q, 10, A} the same? Yes → order doesn't matter

so $\frac{52!}{47!}$ over counts the hands

Any ordering or permutation of {A, K, Q, J, 10} is the same hand.

The number of ordering for {A, K, Q, J, 10} = 5!

For any possible, $\frac{52!}{47!}$, we count that hand 5! times

Divide by 5! ⇒ $\frac{52!}{47!\,5!}$

2) Second Rule of countin

If the order doesn't matter count ordered objects and then divid by number of orderings.

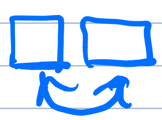
Going back to Poker hands:

How many Poker hands? ($\underline{5}$ cards from $\underline{52}$)

Equivalent to: choose 5 cards from 52 cards (order doesn't matter)

Examples:

choose $\underline{2}$ out of n: $\frac{n \times (n-1)}{2} = \frac{n!}{2!\,(n-2)!}$



choose $\underline{3}$ out of n: $\frac{n \times (n-1) \times (n-2)}{3!} = \frac{n!}{3!\,(n-3)!}$

$3 \times 2 \times 1 = 3!$

choose $\underline{k}$ out of $\underline{n}$: $\frac{n!}{k!\,(n-k)!} \equiv \binom{n}{k}$

$\binom{n}{k}$: "n choose k"

To Summarize:

First rule: $n_1 \times n_2 \times \cdots \times n_k$. Product rule

Second rule: when order doesn't matter divide...

More examples:

ordering of ANAGRAM?

Let's assume A's are distinguishable: $7!$

ANAGRAM, ANAGRAM, ...

$3 \times 2 \times 1 \Rightarrow 3!$ $\Rightarrow$ $\frac{7!}{3!}$

How many ordering of DOG? $3 \times 2 \times 1 = 3!$

CAT

How many ordering of MISSISSIPPI

Assume letters are different: $11!$

4 S's can be Permuted $4!$ times $\Rightarrow \frac{11!}{4!}$

4 I's can be Permuted $4!$ times $\Rightarrow \frac{11!}{4!\,4!}$

2 P's can be Permuted $2!$ times $\Rightarrow \frac{11!}{4!\,4!\,2!}$

3) Sum Rule

Example: Assume we have two indistinguishable jokers in 54 card deck.

How many 5 card hands?

Sum rule: can sum over disjoint Possibilities

3 disjoint Possibilities $\begin{cases} \text{no joker} \quad \text{or} \quad \text{1 joker} \quad \text{or} \quad \text{2 jokers} \\ \binom{52}{5} + \binom{52}{4} + \binom{52}{3} \end{cases}$

Assume we have two distinguishable jokers in 54 card deck.

How many 5 card hands? $\binom{54}{5}$

3 disjoint possibilities = $\left\{ \begin{array}{l} \text{no joker} \mid 1 \text{ joker} \mid 2 \text{ jokers} \\ \binom{52}{5} + 2\binom{52}{4} + \binom{52}{3} \end{array} \right.$

Can we count differently?

$$\binom{54}{5} = \binom{52}{5} + 2\binom{52}{4} + \binom{52}{3}$$

why is this correct? Because we showed that we can count the number of hands by two different approach.

This kind of proof is called combinatorial proof.

4) Combinatorial Proofs:

Theorem: $\binom{n}{k} = \binom{n}{n-k}$

Proof: Come up with a counting story for each side of the equation.

LHS: How many subsets of siz k $\binom{n}{k}$

RHS: Hom many subsets that does not include n-k of the object.

Choose a subset of size n-k objects to not take.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

[Pascal's rule]

LHS: How many size k subsets of n+1 $\binom{n+1}{k}$

RHS: How many size k subset with and without first element

① choose first element

need to choose k−1 more from n

$\binom{n}{k-1}$

② don't choose first element

need to choose k from n

$\binom{n}{k}$

$\hookrightarrow \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$

Theorem: $\binom{n}{k} = \underbrace{\binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k-1}{k-1}}$

LHS: n choose k

RHS: Consider size k subset with i^{th} element is the firs element chosen.

$\{1, 2, \ldots, i, \underbrace{\ldots, n}\}$

choose k-1 from n-i

must choose k-1 elements from n-i remaining. $\Rightarrow \binom{n-i}{k-1}$.

Add them up to get the total number of size k $\Rightarrow \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k-1}{k-1}$

n-(k-1)

So $\binom{n}{k} = \sum_{i=1}^{n-(k-1)} \binom{n-i}{k-1}$.

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

LHS:

RHS: