Counting 2 CS 70, July 15 2020

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1) Combinatorial Theorem

2) Simple Inclusion and Exclusion

3) Inclusion and Exclusion

4) Derangements

5) Sampling

6) Star and Bars

$$2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$$

LHS: $\overbrace{2 \times 2 \times \cdots \times 2}^{n} = 2^n$, The number of subsets of $\{1, \ldots, n\}$

RHS:

subset siz	0 or 1 or 2 ...... or n
	$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

1) Combinatorial Theorem:

$$\underline{(x+y)^n} = \underbrace{x^n\binom{n}{0} + x^{n-1}y\binom{n}{1} + \cdots + \binom{n}{n}y^n}$$

How to distribute $\underline{n}$ ball among $\underline{x}$ red bins and y blue bins?

LHS: $(x+y)\times(x+y)\times \cdots \times(x+y)$

Make subsequent choices:

$$\left.\begin{matrix} x+y \\ x+y \\ \vdots \\ x+y \end{matrix}\right\} \Rightarrow (x+y)^n \text{ Possibilities.}$$

RHS:

$i = 0 \to n$

n balls in red, 0 ball in blue	or n-1 balls in red, 1 ball in blue	...	i balls in red, n-i balls in blue
$\binom{n}{0} y^0 \times x^n$	$\binom{n}{1} y^1 \times x^{n-1}$		$\binom{n}{n-i} y^{n-i} \times x^i$

$$\underbrace{\binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n}$$

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} \underbrace{x^{n-i}y^i}$$

$x=1, y=1 \Rightarrow 2^n = \sum_{i=0}^{n}\binom{n}{i}$

$x=1, y=-1 \Rightarrow$ $\boxed{0 = \sum_{i=0}^{n}\binom{n}{i}(-1)^i}$

2) Simple Inclusion / Exclusion:

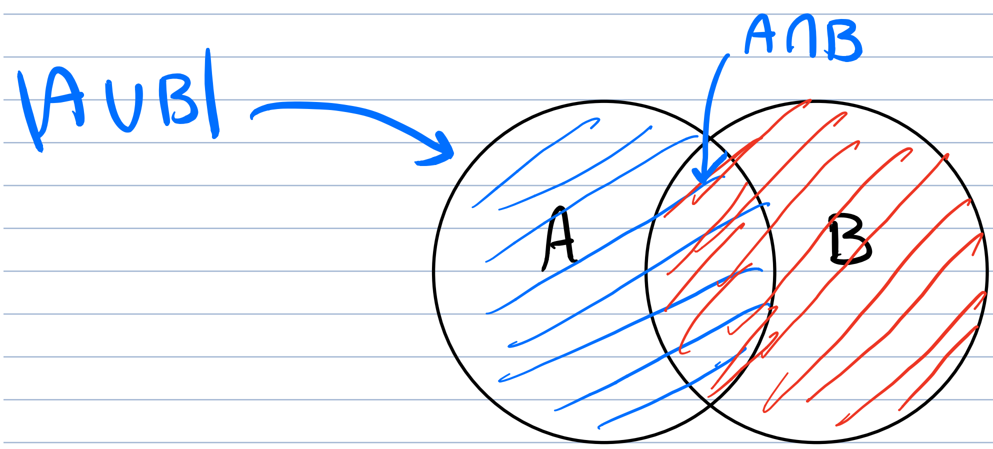
Sum Rule: For disjoint sets A and B ($|A \cap B| = 0$) to count the number of elements of $A \cup B$ we have

$$|A \cup B| = \underline{|A|} + \underline{|B|}$$

when A and B have common element:

Inclusion-Exclusion Rule:

$$\underline{|A \cup B| = |A| + |B|} - |A \cap B|$$



$\rightarrow |A| + |B|$ Elements in $|A \cap B|$ are counted twice

$\Rightarrow$ subtract $|A \cap B|$

Example: How many 10-digit Phone numbers have 5 as their first or second digit?

A = numbers with 5 as first digit, $|A| = 10^9$

B = numbers with 5 as second digit, $|B| = 10^9$

A   5   _   _ _ _ _ _ _ _ _ $\to 10^9$

B   5

$A \cap B$ = numbers with 5 as first and second digit

$|A \cap B| = 10^8$

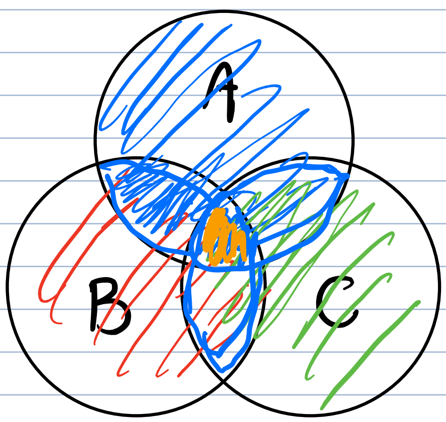
$|A \cup B| = |A| + |B| - |A \cap B| = 10^9 + 10^9 - 10^8.$

- Three way inclusion-Exclusion Rule:

Set A, B, C

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C|$

$- |B \cap C| + |A \cap B \cap C|$



3) Inclusion - Exclusion Principle:

Sets $A_1, \dots, A_n$

$$\left| \bigcup_i A_i \right| = \sum_i |A_i| - \sum_{i_1 \neq j_2} |A_{i_1} \cap A_{j_2}| + \dots$$

$$+ (-1)^{n-1} \sum_{i_1 \neq i_2 \neq \dots \neq i_n} |A_{i_1} \cap \dots A_{i_n}| .$$

4) Derangements:

Permutations of $1, \dots, n$? $n!$

How many permutations where no item in its proper place or fixed points (Derangements)?

Example: Number of derangements 123?

Derangements?
$$\begin{cases} 123 & \text{No!} \\ 213 & \text{No!} \\ 231 & \text{Yes} \end{cases}$$

We can count the complement: count permutations with at least one fixed points.

A_i = "Permutation where i is fixed points"

$i = 1, 2, 3$

Complement "Permutations with at least one fixed points." $= |A_1 \cup A_2 \cup A_3|$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$= 2! + 2! + 2! - 1 - 1 - 1 + 1$$

$$= 4$$

Subtract this from the total Permutations

\# Derangement $= 3! - 4 = 2$ Permutations

For n items:

Permutations with at least one fixed point?

$\boxed{0! = 1}$

$$\left|\bigcup_i A_i\right| = \sum_i |A_i| - \sum_{i_1 \neq j_2} |A_{i_1} \cap A_{j_2}| + \dots + (-1)^{n-1} \sum_{i_1 \neq i_2 \neq \dots \neq i_n} |A_{i_1} \cap \dots A_{i_n}|.$$

$$\underbrace{\binom{n}{1}(n-1)! \quad \binom{n}{2}(n-2)! \quad \cdots + (-1)^{n-1}\binom{n}{n}0!}$$

# of derangements $= \underline{n!} - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)!$

$+ \cdots + (-1)^n \binom{n}{n}$

$= n! - \frac{n!}{\cancel{(n-1)!}\,1!}\cancel{(n-1)!} + \frac{n!}{\cancel{(n-2)!}\,2!}\cancel{(n-2)!} + \cdots + \frac{n!}{n!}(-1)^n$

$= n! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$

$= n! \times \underbrace{\sum_{i=0}^{n} \frac{(-1)^i}{i!}}$ $\Big\}$ $n! \times \frac{1}{e} = \underline{0.37\,n!}$ $\quad n \to \infty$

Roughly $\underline{0.37}$ of the Permutations are derangements!

5) Sampling:

Assume sample $\underline{k}$ items out of $\underline{n}$:

- Without replacement
  - order matters: $n \times (n-1) \times (n-2) \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$
  - order does not matter:

second rule: divid by number of orders $k!$

$= \frac{n!}{(n-k)!\,k!}$

- with replacement:
  - order matters: $n \times n \times \cdots \times n = n^k$
  - order doesn't matters: can we use second rule?

Problem? depends on how many of each iten we choose.

For chosen string ABCD $\longrightarrow$ 4! orderings

" " " $\underbrace{AA}$CD $\rightarrow$ $\frac{4!}{2!}$ ordering

- Different number of ordered elements map to each unordered.

Another example:

How many ways can Alice and Bob split $5?

For each of 5 dollars pick Alice or Bob

2^5 and divid out order

A: 5	B: 0 :	(A, A, A, A, A)	1
A: 4	B: 1 :	(A, A, A, A, B), (A, A, A, B, A)	5
A: 3	B: 2 :	(A, A, A, B, B), ...	10
A: 2	B: 3 :	(A, A, B, B, B), ...	10
A: 1	B: 4 :	(A, B, B, B, B), ...	5
A: 0	B: 5 :	(B, B, B, B, B)	1

↳ second rule of counting is no good here!

Another example!

How ma ways can Alice, Bob, and Eve split $5?

Idea: separat Alice's dollars from Bob's and then Bob's from Eve's.

Assume dollars are 5 stars: * * * * *

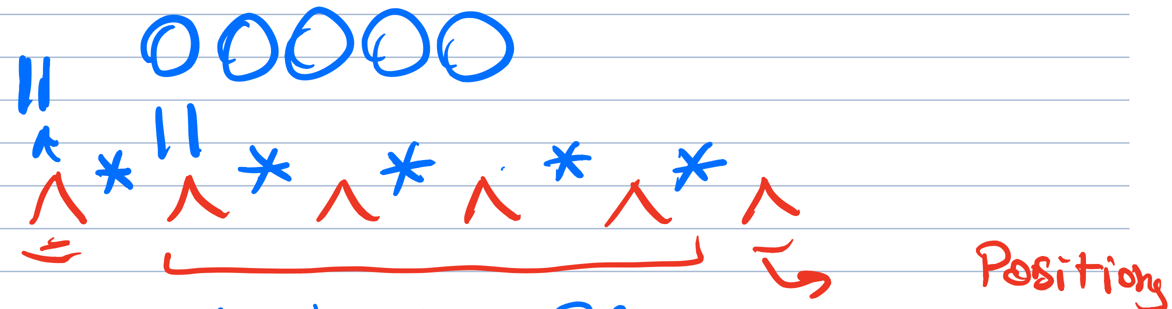
split: Alice: 2, Bob: 1, Eve: 2 ⟺ Stars and bars **|*|**

Split: Alice: 0, Bob: 2, Eve: 3 ⟺ Stars and bars |**|***

A B E

Zeroth Rule Counting: If there is a one-to-one mapping between two set they have the same size.

So we can ask: How many different sequence of 5 stars and 2 bars are there



7 Positions in which to Place 2 bars.

$\Rightarrow$ 7 choose 2: $\binom{7}{2}$ way to do this

$\binom{7}{2}$ ways to split \$5 among 3 People

6) Star and Bars:

ways to split k dollars among n people.

$\Downarrow$ k from n with replacement where order doesn't matter

Correspondence: $n-1$ bars to split the k stars

$* * | \cdots | * | \cdots |$

$n+k-1$ positions from which to choose

$n-1$ bar positions:

$$\binom{n+k-1}{n-1}.$$