Introduction to Probability July, 16, 2020

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1) Key Points
2) Random Experiments
3) Probability Space
4) Complement of an Event

★ Problem of the day: How do you place $\underline{n}$ good candies and $\underline{n}$ bad candies in two boxes such that if you choose a box at random and take out a candy at random, it better be good?!

1) Key Points:

- Uncertainty does not mean "nothing is known"
- How to best make decision under uncertainty?
  - Buy stocks
  - Detect signals (transmitted bits, radar, ...)
  - Control systems (Internet, airplane, robots)
- How to best use 'artificial' uncertainty?
  - Play games of chance
  - Design randomized algorithms.
- Probability knowledge about uncertainty
  - models knowledge about uncertainty.
  - Discovers best way to use that knowledge in making decisitions.

Uncertainty: vague, fuzzy, confusing, scary!
Probability: A precise, unambigouse way of thinking about uncertainty

Our mission: Help you think clearly about uncertainty!

2) Random Experiments

Possibilities

Flip one Fair Coin

H T

What do we mean by the likelihood of tails is 50%?

Two intepretations:

- single coin flip: 50% chance of 'tails' [subjective]

  willingnes to bet on the outcome of single flip

- Many coin flips: About half yield 'tails' [frequentist]

  Makes sense only many flips.

Question: why does the fraction of tails converge to the same value every time?

Statistical Regularity!

The Probability model.

- A set of outcomes: $\{H, T\}$

- H: 50%
- T: 50%

Random Experiment:

Flip one Unfair Coin

- H: 45%
- T: 55%

- Possible outcomes: Heads(H) and Tails (T)
- Likelihoods: H: $P \in (0,1)$, T: $1-P$

- H: P
- T: $1-P$

- Frequentist Interpretation:

  Flip this coin many times $\Rightarrow$ A fraction $(1-P)$ of tails.

- Question: How can one figure out P?

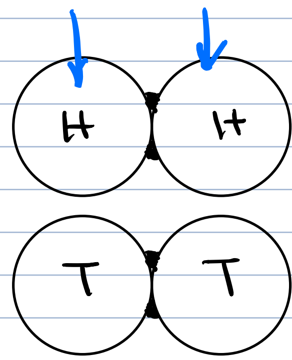
  Flip many times $\rightarrow$ Statistical Regularity!

Flip Two Fair Coins:

- Possible outcomes: $\{HH, TH, HT, TT\} = \{H,T\}^2$
- Note: $A \times B := \{(a,b) \mid a \in A,\ b \in B\}$, $A^2 = A \times A$
- Likelihoods: HH: $\frac{1}{4}$, TH: $\frac{1}{4}$, HT: $\frac{1}{4}$, TT: $\frac{1}{4}$

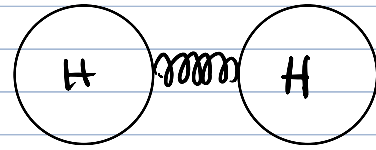
Flip Glued Coins:

- Note: Coins are glued so that they show the same face.



- Possible outcomes: $\{HH, TT\}$
- Likelihoods: HH: $\frac{1}{2}$, TT: $\frac{1}{2}$

Flip two coins attached by a spring:



- Possible outcomes:

  $\{HH, HT, TH, TT\}$

- Likelihoods: $HH: 0.4, HT: 0.1, TH: 0.1, TT: 0.4$

Flipping n times:

Flip a fair coin n times:

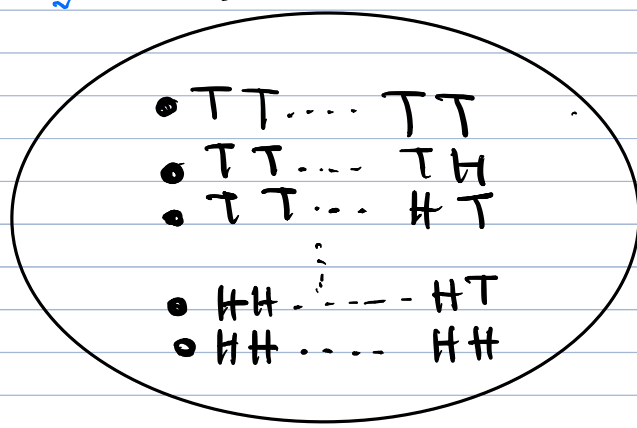
- Possible outcomes: $\{TT\ldots T, TT\ldots H, \ldots, HH\ldots H\} = \{H, T\}^n$

  Thus, $2 \times 2 \times \cdots \times 2 = 2^n$ possible outcomes

- Note: $\{TT\ldots T, TT\ldots H, \ldots, HH\ldots H\} \equiv \{H, T\}^n$

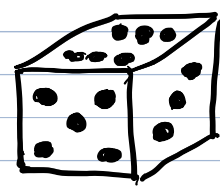
$$A^n := \{(a_1, \ldots, a_n) \mid a_1 \in A, \ldots, a_n \in A\}, \quad |A^n| = |A|^n.$$

- Likelihoods: $1/2^n$ each.



Roll a Die

Roll a balanced 6-sided die:



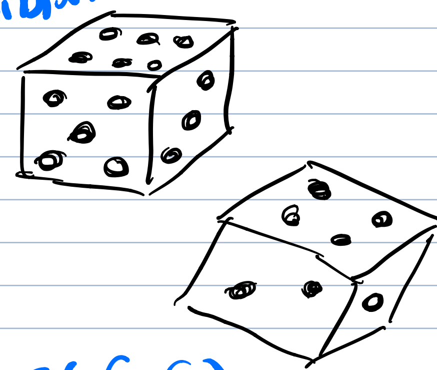
- Possible outcomes: $\{1, 2, 3, 4, 5, 6\}$
- Likelihoods: $\frac{1}{6}$ for each.

Roll two Dice:

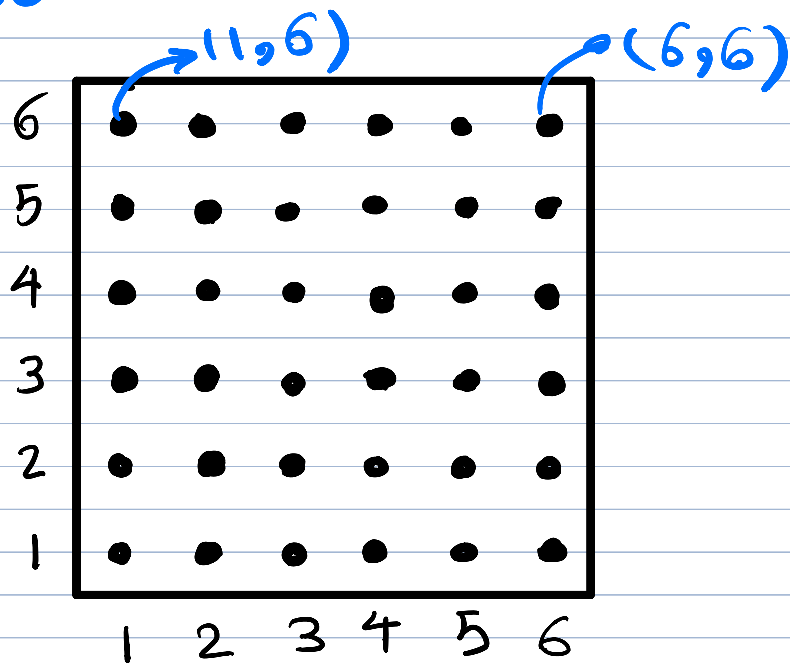
Roll a balanced 6-sided die twice:

- Possible outcomes: $\rightarrow 6^2 = 36$ Possibilities

$\{1,2,3,4,5,6\}^2 = \{(a,b) \mid 1 \leq a,b \leq 6\}$



- Likelihoods: $\frac{1}{36}$ for each



3) Probability Space:

1. A random experiment:
   (a) Flip a biased coin.
   (b) Flip a two fair coins
   (c) Deal a Poker hand

2. A Set of Possible outcome: Ω

(a) $\Omega = \{H, T\} \rightarrow |\Omega| = 2$

(b) $\Omega = \{HH, HT, TH, TT\} \Rightarrow |\Omega| = 4$

(c) $\Omega = \{\underline{A\spadesuit}\ \underline{A\diamondsuit}\ \underline{A\clubsuit}\ \underline{A\heartsuit}\ \underline{K\spadesuit}, \underline{A\spadesuit}\ \underline{A\diamondsuit}\ \underline{A\clubsuit}\ \underline{A\heartsuit}\ \underline{Q\spadesuit}, \dots\}$

$|\Omega| = \binom{52}{5}$

3. Assign a Probability to each outcome:

$Pr: \Omega \rightarrow [0, 1]$

(a) $Pr[H] = p$, $Pr[T] = 1-p$

(b) $Pr[HH] = Pr[HT] = Pr[TH] = Pr[TT] = \frac{1}{4}$

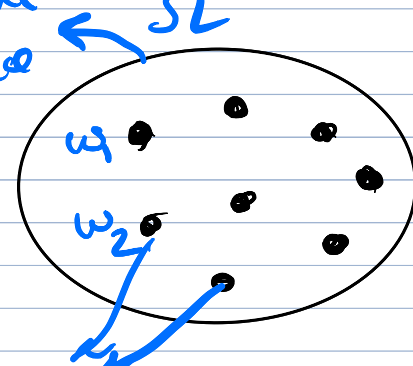
(c) $Pr[\underline{A\spadesuit}\ \underline{A\diamondsuit}\ \underline{A\clubsuit}\ \underline{A\heartsuit}\ \underline{K\spadesuit}] = \frac{1}{\binom{52}{5}}$

Probability Space: Formalism

- Ω is the sample space
- $w \in \Omega$ is a sample point (Also called outcome)
- Sample point w has Probability $Pr[w]$

where (1) $0 \leq Pr[w] \leq 1$, $\sum_{w \in \Omega} Pr[w] = 1$

- In uniform Probability space each out come ω is equally Probable

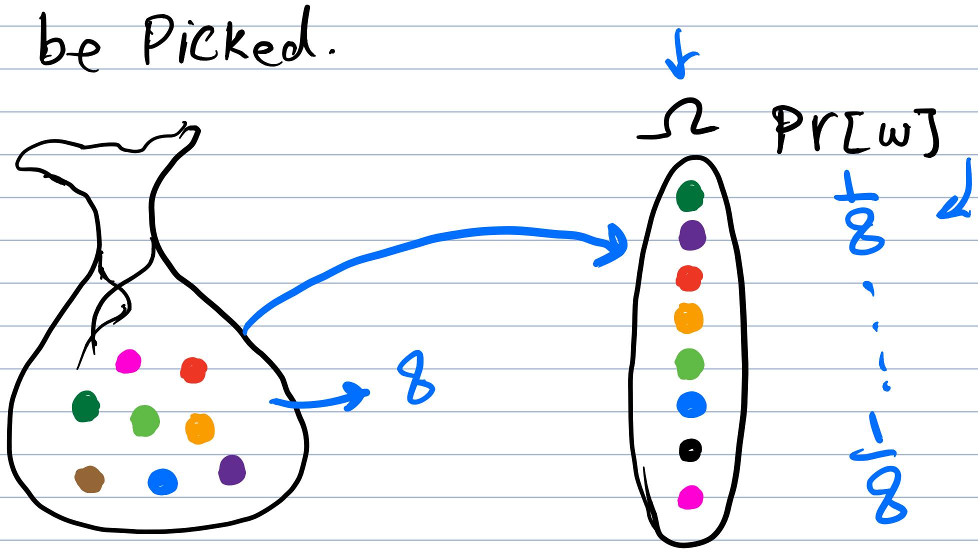


$$Pr[\omega] = \frac{1}{|\Omega|} \text{ for all } \omega \in \Omega.$$

Examples)

A simple model of a uniform Probability space:

A bag of identical balls, except for their color. If the bag is well shaken, every ball is equally likely to be Picked.

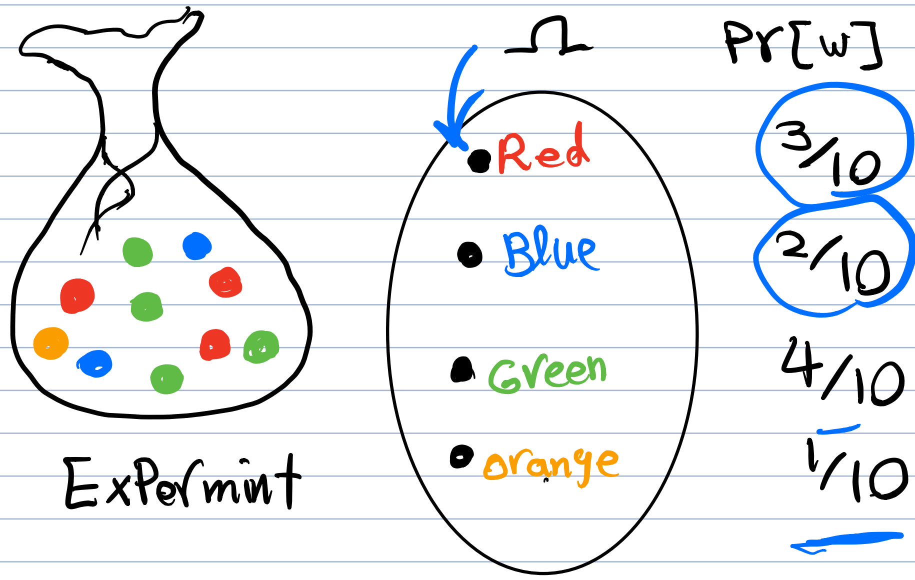


Experiment   Probability model

Ω = { Green, Dark Green, Pink, Purple, Blue, orange, Brown, Red }

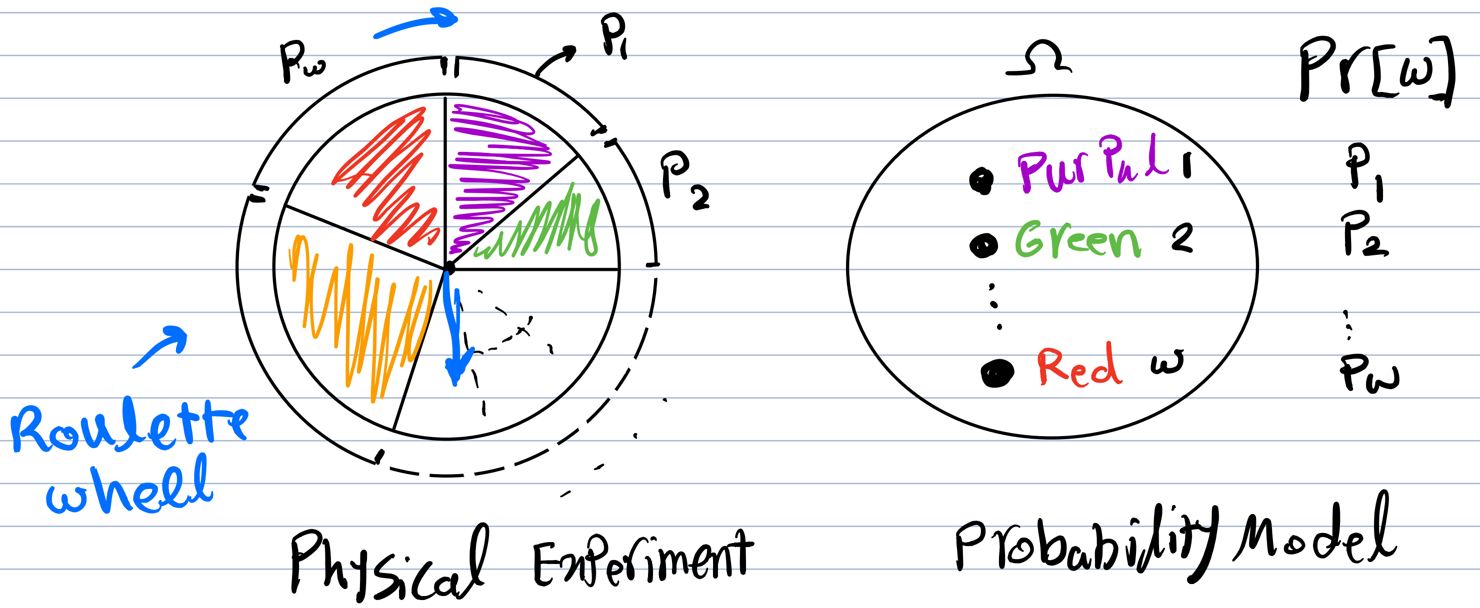
$$Pr[blue] = 1/8.$$

A simple model of a non-uniform Probability space:



$$\Omega = \{Red, Blue, Green, orange\}$$

A General model of non-uniform Probability space:



- The roulette wheel stop in sector ω with Probability p_ω

$$\Omega = \{1, 2, \ldots, N\}, \quad Pr[\omega] = p_\omega$$

An Important Remark

- The random experiment selects one and only one outcome in Ω.

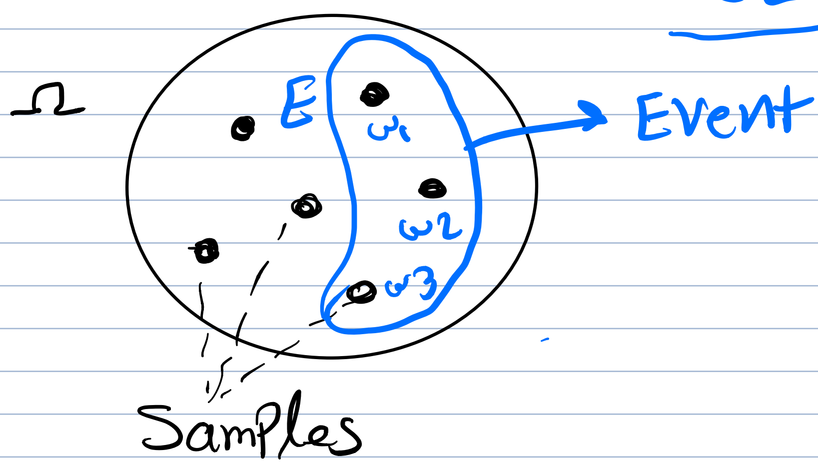
Probability of exactly one heads in two coin flip?

Idea: sum the Probability of all different outcomes that have exactly one H: HT, TH

This leads to a definition:

- An event, E, is a subset of outcomes $E \subseteq \Omega$
- The Probability of E is defined as

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$



Uniform Probability Space: $\Pr[\omega] = \frac{1}{|\Omega|}$

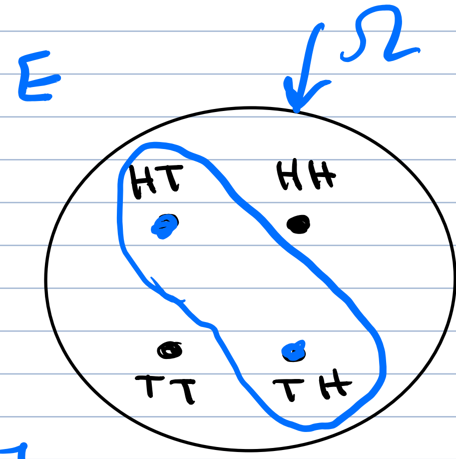
$$\Pr[E] = \frac{|E|}{|\Omega|}$$

Probability of exactly one heads in two coin flip?

- Sample Space, Ω

$\{HH, HT, TH, TT\}$

$Pr[\omega] = \frac{1}{|\Omega|} = \frac{1}{4}$



Event, E, "exactly one heads":

$\{HT, TH\} \Rightarrow Pr[E] = \underbrace{\sum_{\omega \in E} Pr[\omega]}_{} = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}$

- What if the coin is biased? $\boxed{Pr[H] = P}$

$Pr[E] = \sum_{\omega \in E} Pr[\omega] = Pr[HT] + Pr[TH]$

$= \underbrace{P(1-P)} + \underbrace{(1-P)P} = \underline{2P(1-P)}$

Example: <u>10</u> <u>coin</u> tosses

Sample space $\underline{\Omega}$ = Set of <u>10</u> fair coin tosses.

$\Omega = \{H, T\}^{10} = \{0, 1\}^{10}$, $|\Omega| = 2^{10}$

- what is more likely?
  - $\omega_1 = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$
  - $\omega_2 \{1, 0, 1, 1, 0, 0, 0, 0, 1, 1\}$

Both are equally likely $Pr[\omega_1] = Pr[\omega_2] = \frac{1}{|\Omega|}$

$= \frac{1}{2^{10}}$

- what is more likely?

E_1: 10 heads out of 10 tosses.

E_2: 5 heads out of 10 tosses

Answer: E_2

- $Pr[E_1] = \frac{1}{2^{10}}$

- There many sequences with 5 heads out of 10 tosses.

$$Pr[E_2] = \frac{\binom{10}{5}}{2^{10}}$$

4) Complement of an Event:

Remember $\sum_{w \in \Omega} Pr[w] = 1$

$\bar{E}$: is the complement of E.

$$\underbrace{Pr[E] = \sum_{w \in E} Pr[w]}$$

$$Pr[\bar{E}] = \sum_{w \notin E} Pr[w]$$

$$1 = \sum_{w \in \Omega} Pr[w] = \sum_{w \in E} Pr[w] + \sum_{w \notin E} Pr[w] = \underbrace{Pr[E] + Pr[\bar{E}]}$$

$$Pr[E] + Pr[\bar{E}] = 1 \Rightarrow \boxed{Pr[\bar{E}] = 1 - Pr[E]}$$

Note:

Sometimes it is easier to find the complement of E.

Example: Birthday Paradox

What is the probability that at least two people in a group of $\underline{n}$ people have the same birthday?

E: At least two people with the same birthday among n people.

There are 365 days a year.

$$|\Omega| = \underbrace{365 \times \cdots \times 365}_{n} = 365^n$$

E = there can be at least one pair of people with the same birthday.

$$Pr[E] = 1 - Pr[E]$$

E: No two people have the same birthday

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{365 \times 364 \times \cdots \times (365-n+1)}{365^n}$$

$$Pr[E] = 1 - \frac{365 \times 364 \times \cdots \times (365-n+1)}{365^n}$$

Find n such that $Pr[E] \geq 0.5$?

$$n = 23 \Rightarrow Pr[E] \geq 0.5$$