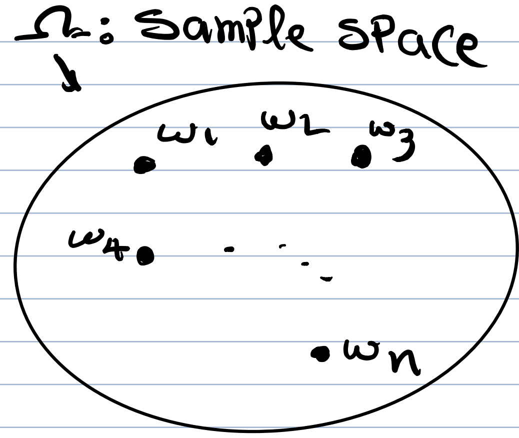
# Conditional Probability

July, 20, 2020

1) The Monty Hall Problem
2) Conditional Probability
3) Bayes' Rule
4) Total Probability Rule.

Last time:

$\omega_i$: Sample Point (outcome)
$i = 1, \dots, n$

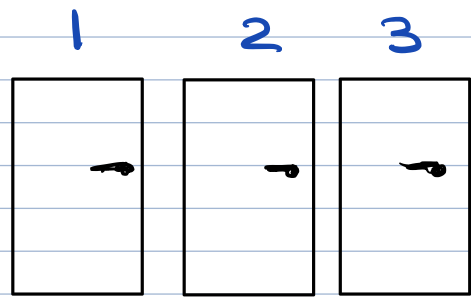



- $0 \leq Pr[\omega_i] \leq 1$
- $\sum_{i=1}^{n} Pr[\omega_i] = 1$

- Events: $E \Rightarrow Pr[E] = \sum_{\omega_i \in E} Pr[\omega_i]$
- The complement of event $E$ : $\bar{E} \Rightarrow Pr[E] = 1 - Pr[\bar{E}]$

1) The Monty Hall Problem:

- There are three doors.
- A car is behind one of the doors!




- Goats behind the other two doors.

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1. The contestant picks a door (but does not open it)
2. Then, Hall's assistant opens one of the other two doors, revealing a goat.
3. The contestant is then given the option of sticking with their current choice or switching to the other unopened door.
4. He/She win the car if and only if their chosen door is the correct one.

Question: Does the contestant have a better chance of winning if he/she switches door?

Assume the prize is equally likely to be behind any of the three doors.

Sample Space

c: car

g: goat

$\Omega = \{cgg, gcg, ggc\}$

| | 1 | 2 | 3 | Pr |
|---|---|---|---|---|
| $\omega_1$: | c | g | g | 1/3 |
| $\omega_2$: | g | c | g | 1/3 |
| $\omega_3$: | g | g | c | 1/3 |

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WLOG we assume $\omega_1$ happens

| 1 | 2 | 3 |
|---|---|---|
| c | g | g |

$\Omega' = \{1, 2, 3\}$

$Pr[1] = Pr[2] = Pr[3] = \frac{1}{3}$

$E_1$: winning by switching

$E_1$:

| chosen | opened | switch |
|---|---|---|
| 1 | 2, 3 | 3, 2 |
| 2 | 3 | 1 |
| 3 | 2 | 1 |

$$Pr[E_1] = Pr[1 \to 2] + Pr[1 \to 3] + Pr[2 \to 1] + Pr[3 \to 1] = 0 + 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$Pr[1 \to 2] = \frac{1}{3} \times 0 = 0$$

$$\boxed{Pr[E_1] = \frac{2}{3}}$$

$Pr[1 \to 3] = \frac{1}{3} \times 0 = 0$

$Pr[2 \to 1] = \frac{1}{3} \times 1 = \frac{1}{3}$

$Pr[3 \to 1] = \frac{1}{3} \times 1 = \frac{1}{3}$

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$E_2$: winning by not switching

| | chosen | opened | stick to |
|---|---|---|---|
| $E_2$: | 1 | 2,3 | 1 |
| | 2 | 3 | 2 |
| | 3 | 2 | 3 |

$Pr[E_2] = Pr[1 \to 1] + Pr[2 \to 2] + Pr[3 \to 3]$

$= \frac{1}{3} + 0 + 0 = \frac{1}{3} \Rightarrow \boxed{Pr[E_2] = \frac{1}{3}}$

$Pr[1 \to 1] = \frac{1}{3} \times 1 = \frac{1}{3}$

$Pr[2 \to 2] = \frac{1}{3} \times 0 = 0$

$Pr[3 \to 3] = \frac{1}{3} \times 0 = 0$

We have a better chance of winning if we use switching strategy.

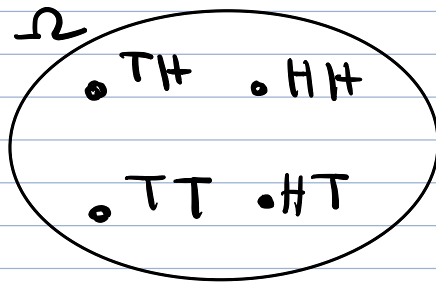
## 2) Conditional Probability:

Examples: Two coin flips. First flip is heads.
Probability of two heads?

$\Omega = \{\underline{HH}, \underline{HT}, \underline{TH}, \underline{TT}\}$, uniform probability space.

Event A = first flip is heads: $A = \{HH, HT\}$

New sample space




Event B = two heads $= \{HH\}$

The probability of two heads if the first flip is heads.

The probability of B given A is $Pr[B|A] = \frac{1}{2}$

Example: Two coin flips. At least one of the flips is heads. Probability of two heads?

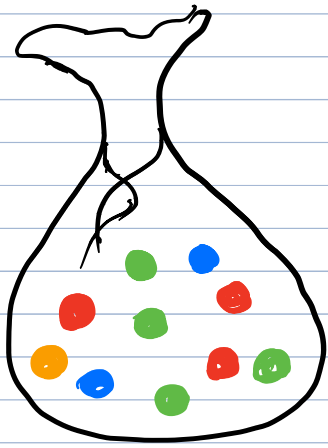
Event $A = \{HT, TH, HH\}$

Event $B = \{HH\}$

$\Rightarrow$ The probability of B given A

$\Rightarrow Pr[B|A] = \frac{1}{3}$

# A non-uniform example



Expermint

| $\Omega$ | $Pr[w]$ |
|---|---|
| Red | 3/10 |
| Blue | 2/10 |
| Green | 4/10 |
| orange | 1/10 |

$$\Omega = \{Red, Blue, Green, orange\}$$

$$Pr[\underbrace{red}_{B} \mid \underbrace{red \text{ or } green}_{A}] = \frac{3}{7}$$

$A = \{3 \text{ reds}, 4 \text{ greens}\}$

$B = \{3 \text{ reds}\}$

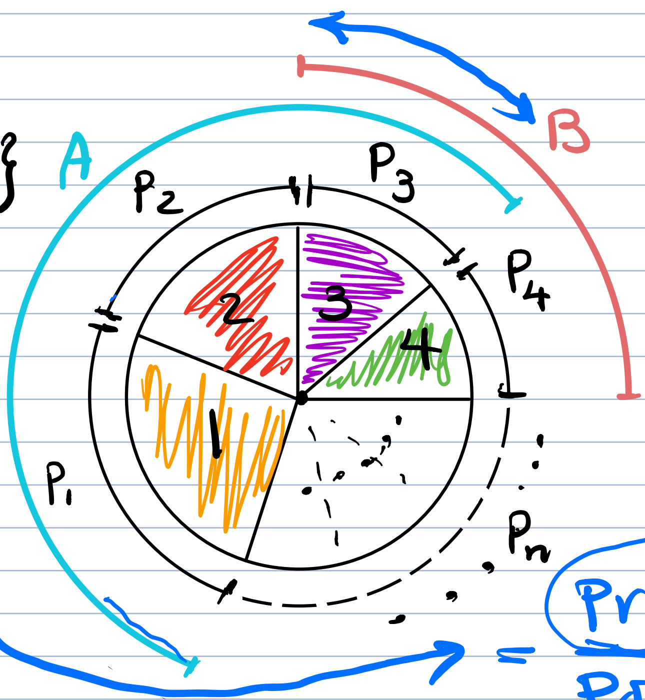
Another example:

Consider $\Omega = \{1, 2, \dots, N\}$

with $Pr[n] = P_n$

Let $A = \{1, 2, 3\}$

$B = \{3, 4\}$

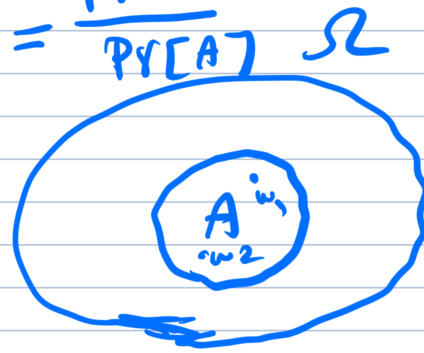



$$Pr[B|A] = \frac{P_3}{P_1 + P_2 + P_3} = \frac{\boxed{Pr[A \cap B]}}{Pr[A]}$$

Joint Probability

Assume sample point $\omega \in A$, then

$$Pr[\omega|A] = ? = \frac{Pr[\omega \cap A]}{Pr[A]} = \frac{Pr[\omega]}{Pr[A]}$$




Then

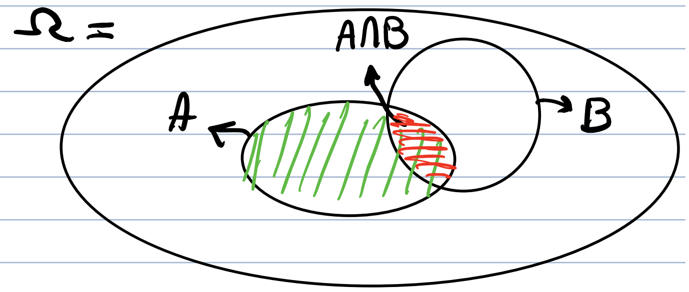
$$Pr[B|A] = ? = \frac{\sum_{\omega \in B} Pr[\omega \cap A]}{Pr[A]} = \frac{Pr[B \cap A]}{Pr[A]}$$

Definition: The conditional probability of B given A is

$$Pr[B|A] = \frac{Pr[B \cap A]}{Pr[A]}$$

A given B

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$$

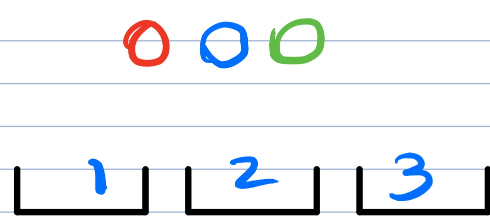



One more example:

Suppose I toss 3 balls into 3 bins, one at the time

A = "1st bin is empty"; B = "2nd bin is empty"

What is $Pr[A|B]$?




$\Omega = \{1,2,3\}^3 \rightarrow |\Omega| = 3^3 = 27$

$A = \{2,3\}^3$, $B = \{1,3\}^3$, $A \cap B = \{3\}^3$

$|A| = 2^3 = 8$, $|B| = 2^3 = 8$, $|A \cap B| = 1^3 = 1$

$$\Pr[A|B] = \frac{\Pr[A\cap B]}{\Pr[B]} = \frac{\frac{1}{27}}{\frac{8}{27}} = \frac{1}{8}.$$

$$\Pr[B] = \frac{|B|}{|\Omega|} = \frac{8}{27}, \quad \Pr[A\cap B] = \frac{|A\cap B|}{|\Omega|} = \frac{1}{27}$$

Bayesian Inference:

- A way update knowledge after making an observation
- we may have an estimate of Probability of a given event A. → Prior Knowledge
- After event B occures we can update this estimate to $\Pr[A|B]$
- In this interpretation $\begin{cases} \Pr[A] : \text{Prior Probability} \\ \Pr[A|B] : \text{Posterior Probability} \end{cases}$

Example:

There is a new test for a certain disease.

1) when the test applied to an affect Person, the test comes up Positive 90% of cases. and negative in 10% → ("False negative")

2. when applied to a healty Person the test

$\bar{A}$

comes up negative 80% of cases and Positive 20%.. "False Positive"

- suppose that only 5% of the Population has this disease. (Prior) → A

Q. When a random Person is tested Positive, what is the Probability that the Person has the disease? → $Pr[A|B]$

Let's define events:

A: affected

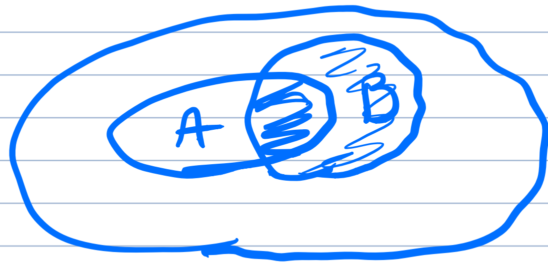
B: Test is Positive.

$Pr[A]=0.05$, $\underline{Pr[B|A]}=0.9$, $Pr[B|\bar{A}]=0.2$

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{Pr[B|A]\,Pr[A]}{Pr[B]}$$

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]} \Rightarrow Pr[A \cap B] = \underline{Pr[B|A]\,Pr[A]}$$

$$Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B]$$



$$\Pr[A \cap B] = \Pr[B \mid A]\Pr[A]$$

$$\Pr[\bar{A} \cap B] = \Pr[B \mid \bar{A}]\Pr[\bar{A}] = \Pr[B \mid \bar{A}](1-\Pr[A])$$

$$\Pr[A \mid B] = \frac{\Pr[B \mid A]\Pr[A]}{\Pr[B \mid A]\Pr[A] + \Pr[B \mid \bar{A}](1-\Pr[A])}$$

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3) Bayes' Rule: $P(A \mid B) = \dfrac{P(B \mid A)\,P(A)}{P(B)}$

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

$$\Pr[B \mid A] = \Pr[A \cap B] / \Pr[A]$$

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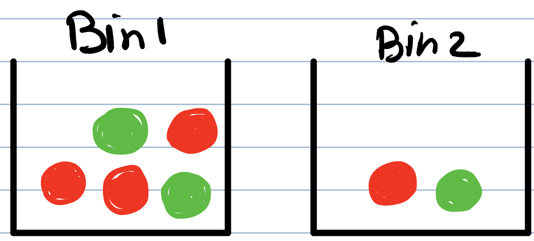
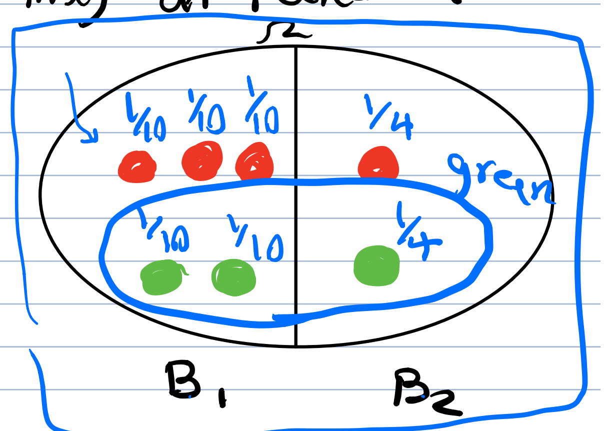
4) Total Probability Rule:

Imagine two bins containing some number of red and green.




- One bin is chosen with equal Probability ⇒ 50%
- Then a ball is drawn uniformly at random.
- what is the probability that we Picked Bin 1 given that a green ball was dran



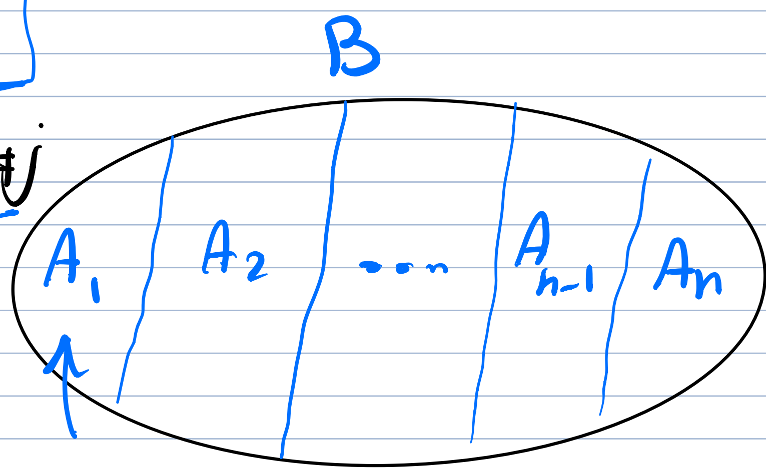
$$Pr[B_1 | \text{green}] = \frac{Pr[\text{green} | B_1]\, Pr[B_1]}{Pr[\text{green}]} = \frac{\frac{1}{5}}{\frac{9}{20}} = \frac{4}{9}$$

$$\begin{aligned} Pr[\text{green}] &= Pr[\text{green} \cap B_1] + Pr[\text{green} \cap B_2] \\ &= Pr[\text{green} | B_1]\, Pr[B_1] + Pr[\text{green} | B_2]\, Pr[B_2] \\ &= \frac{2}{5} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{9}{20} \end{aligned}$$

Definition: Event $B$ is Partitioned into $n$ events $A_1, \dots, A_n$ if

1. $B = \boxed{A_1 \cup A_2 \cup \dots \cup A_n}$
2. $A_i \cap A_j = \emptyset$ for all $i \neq j$

(i.e. $A_1, \dots, A_n$ are mutually exclusive




Then the total Probability is

$$Pr[B] = \sum_{i=1}^{n} Pr[B \cap A_i] = \sum_{i=1}^{n} Pr[B | A_i]\, Pr[A_i]$$

So for the bays rule

$$Pr[A_i / B] = \frac{Pr[B|A_i]\,Pr[A_i]}{\underbrace{Pr[B]}} = \frac{Pr[B|A_i]\,Pr[A_i]}{\sum_{j=1}^{n} Pr[B|A_j]\,Pr[A_j]}$$