Combinations of Events

July, 21, 2020

1) Independence

2) Union of Events (Inclusion- Exclusion)

3) Union Bound

Last time:

- Conditional Probability $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$

- Bayes' Rule $Pr[A|B] = \frac{Pr[B|A]\,Pr[A]}{Pr[B]}$

- Total Probability Rule:

$$\left.\begin{array}{l} \bullet\ \Omega = \bigcup_{i=1}^{n} A_i \\ \bullet\ A_i \cap A_j = \phi \\ \quad \text{for all } i \neq j \end{array}\right\} \Rightarrow Pr[B] = \sum_{i=1}^{n} Pr[B|A_i]\,Pr[A_i]$$

Interested in things like $Pr[\cup_{i=1}^{n} A_i]$ and $Pr[\cap_{i=1}^{n} A_i]$, where A_i are some events and we know $Pr[A_i]$

1) Independent Events.

Definition: Two events A, B in the same Probability space are independen if

$$Pr[A \cap B] = Pr[A]\, Pr[B].$$

Examples:

- when rolling two dice $\Rightarrow |\Omega| = 6^2 = 36$

  A = sum is 7 and B = red die is 1

  A and B are independent. $A = \{16, 25, 34, 43, 52, 61\}$

  $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]\, Pr[B] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

  A = sum is 3 and B = red die is 1

  A and B are not independent $A = \{12, 21\}$

  $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]\, Pr[B] = \frac{2}{36} \times \frac{1}{6}$

- when flipping coins

  A = Coin 1 yields heads, B = coin 2 yields tails

  A and B are idePendent.

  $Pr[A \cap B] = \frac{1}{4}$ $Pr[A]\, Pr[B] = \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

- when throwing 3 balls into 3 bins

A = bin 1 is empty , B = bin 2 is empty

A and B are ~~not-independent~~

$Pr[A \cap B] = \frac{1}{27}$ $\qquad Pr[A]Pr[B] = \frac{8}{27} \times \frac{8}{27}$

Fact: Two events A and B are independent if and only if

$Pr[A|B] = Pr[A]$ → the Probability of A is not affect by B.

if $Pr[A|B] = Pr[A]$ ⇒ A and B independent

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = Pr[A]$$

$$\Rightarrow Pr[A \cap B] = Pr[A]\,Pr[B].$$

$$Pr[A|B]\,Pr[B] = Pr[A]\,Pr[B]$$

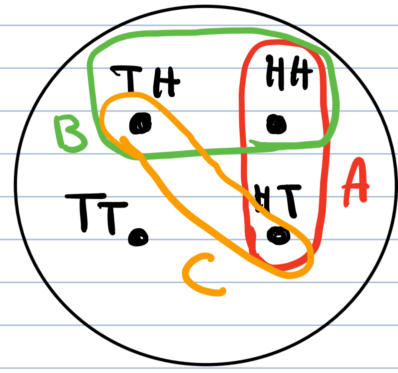
$$\Rightarrow Pr[A|B] = Pr[A]$$

Pairwise Independence:

Flip two coins. Let

- A = 'First coin is H' $\quad Pr[A] = \frac{1}{2}$
- B = 'Second coin is H' $\quad Pr[B] = \frac{1}{2}$
- C = 'The two coins are different' $\quad Pr[C] = \frac{1}{2}$

A and B are independent
B and C are independent
A and C are independent



$\underbrace{A \cap B}$ and C are $Pr[(A\cap B) \cap C] = \frac{0}{4} = 0$

$\downarrow$ if independent $= \underbrace{Pr[A\cap B]}_{\frac{1}{4}} \times \underbrace{Pr[C]}_{\frac{1}{2}}$

not independent. $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

They are not completely independent.

Definition: (mutual independence)

Events $A_1, \dots, A_n$ are mutually independent if for every subset $I \subseteq \{1, \dots, n\}$ with $|I| \geq 2$:

$$Pr[\cap_{i\in I} A_i] = \prod_{\underline{i\in I}} Pr[A_i]$$

How many constraints?

$S = \{1, \dots, n\}$ $\qquad |P(S)| = \underline{2^n}$

How many I ?

I = subsets with size ≥ 2

$\# \text{ constraints} = 2^n \underbrace{-1}-n$

Equivalently:

Events $A_1, \dots, A_n$ are mutually independent if for all $B_i \in \{A_i, \bar{A}_i\}$, $i = 1, \dots, n$.

$$\Pr[B_1 \cap \cdots \cap B_n] = \prod_{i=1}^{n} \Pr[B_i]$$

How many constraints? $\underbrace{\{A_1, \bar{A}_1\} \times \{A_2, \bar{A}_2\} \times \cdots \times \{A_n, \bar{A}_n\}}$

$\# = 2^n$

The extra constraints are redundant.

- Intersection of Events

Recall $\Pr[B \mid A] = \dfrac{\Pr[A \cap B]}{\Pr[A]}$

Then $\Pr[A \cap B] = \Pr[A]\,\Pr[B \mid A]$

Consequently:

$$\Pr[A \cap B \cap C] = \Pr[\underbrace{(A \cap B)}_{D} \cap C]$$

$$= \Pr[A \cap B] \times \Pr[C \mid \underbrace{A \cap B}_{D}]$$

$$= \Pr[A]\,\Pr[B \mid A]\,\Pr[C \mid A \cap B]$$

Theorem: Product Rule

Let $A_1, A_2, \ldots, A_n$ be events. Then

$$\Pr[A_1 \cap \ldots \cap A_n] = \Pr[A_1]\Pr[A_2 | A_1] \cdots \Pr[A_n | A_1 \cap \cdots \cap A_{n-1}]$$

Proof: By induction

It holds for $n=2$.

Assume that this holds for some k. Then

$$\Pr[\underbrace{A_1 \cap \ldots \cap A_k}_{A'} \cap A_{k+1}] = \Pr[A'] \times \Pr[A_{k+1} | A']$$

$$= \Pr[A_1 \cap \ldots \cap A_k] \Pr[A_{k+1} | A_1 \cap \ldots \cap A_k]$$

from I.H. $= \Pr[A_1]\Pr[A_2|A_1] \cdots \times \Pr[A_k | A_1 \cap \cdots A_{k-1}]$
$\times \Pr[A_{k+1} | A_1 \cap \ldots \cap A_k]$

Examples:

Toss a biased coin, with probability p, three times.

A = All three tosses are heads.

$A = A_1 \cap A_2 \cap A_3$, A_i = the i^{th} toss comes up head.

$$\Pr[A] = \Pr[A_1 \cap A_2 \cap A_3] = \Pr[A_1]\Pr[A_2|A_1] \times \Pr[A_3 | A_1 \cap A_2]$$

$$= \Pr[A_1]\Pr[A_2]\Pr[A_3] = p \times p \times p = p^3.$$

The Probability of any sequence of n tosses containing K heads and $n-K$ tails is $p^K \times (1-p)^{n-K}$

Example: Balls in bins

Throw m ball into $n \gg m$ bins, one at the time.

Theorem: $\Pr[\text{no collision}] \approx \exp\left(-\frac{m^2}{2n}\right)$

for large enough n.

A_i = no collision when i^{th} ball is placed in a bin

$$\Pr[A_i \mid A_1 \cap \dots \cap A_{i-1}] = 1 - \frac{i-1}{n}$$

Then: A = no collision $= A_1 \cap \dots \cap A_m$

$$\Pr[A_1 \cap \dots \cap A_m] = \Pr[A_1] \Pr[A_2 \mid A_1] \dots \times \Pr[A_m \mid A_1 \cap \dots A_{m-1}]$$

$$\Pr[\text{no collision}] = 1 \times \left(1 - \frac{1}{n}\right) \times \dots \times \left(1 - \frac{m-1}{n}\right)$$

$$\ln \Pr = \sum_{k=1}^{m-1} \ln\left(1 - \frac{k}{n}\right) \approx \sum_{k=1}^{m-1} \left(-\frac{k}{n}\right)$$

$$\begin{cases} n \to \infty \Rightarrow \frac{k}{n} \to 0 \\ \underline{\ln(1-\varepsilon) \approx -\varepsilon} \quad \text{for } |\varepsilon| \ll 1 \end{cases} \qquad = \frac{-1}{n}\sum_{k=1}^{m-1} k = \frac{-1}{n}\,\frac{m(m-1)}{2} \approx \frac{-m^2}{2n}$$

$$\underbrace{\ln Pr \approx -\frac{m^2}{2n}}\Rightarrow \boxed{Pr = \exp\left(-\frac{m^2}{2n}\right)} \to e^{-\frac{m^2}{2n}}$$

2) Union of Events:

Theorem:

(a) For some events A and B

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$$

(b) Let $A_1, \ldots, A_n$ be some events, where $n \geq 2$

$$Pr[\underline{A_1 \cup \ldots \cup A_n}] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\};\ |S|=k} Pr\left[\cap_{i \in S} A_i\right]$$

or

$$Pr\left[\cup_{i=1}^n A_i\right] = \sum_{i=1}^{n} Pr[A_i] - \sum_{i<j} Pr[A_i \cap A_j] + \sum_{i<j<k} Pr[A_i \cap A_j \cap A_k] - \cdots + (-1)^{n-1} Pr\left[\cap_{i=1}^n A_i\right]$$

$\downarrow$

Principle of Inclusion-Exclusion.

Example:

- Pick a number from $\{1,2,3,4,5,6\}$
- Three dice are thrown.

what is the Probability that your number comes up on at least one of the dice so you win?

A_i = my number comes up on die i

A = winning = $A_1 \cup A_2 \cup A_3$

$$\Pr[A] = \Pr[A_1 \cup A_2 \cup A_3]$$
$$= \Pr[A_1] + \Pr[A_2] + \Pr[A_3]$$
$$- \Pr[A_1 \cap A_2] - \Pr[A_2 \cap A_3] - \Pr[A_1 \cap A_3]$$
$$+ \Pr[A_1 \cap A_2 \cap A_3]$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{6} \times \frac{1}{6} - \frac{1}{6} \times \frac{1}{6} - \frac{1}{6} \times \frac{1}{6}$$
$$+ \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \ldots$$

- when n is large, the Inclusion-Exclusion formula is essentially useless because it involves computing the Probability every non-empty subset of $\{A_1, ..., A_n\}$: $2^n - 1$ terms

3) Union Bound:

However, in many situations we can get a long way by just looking at the first term:

1. (mutually exclusive events)

If the events $A_1, ..., A_n$ are mutually exclusive (i.e. $A_i \cap A_j = \emptyset$ for all $i \neq j$)

$$Pr\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} Pr[A_i]$$

2. (Union bound) Let $A_1, ..., A_n$ be events in some Probability space. Then, for all $n \in \mathbb{Z}^+$

$$Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq \underbrace{\sum_{i=1}^{n} Pr[A_i]}$$

Adding $Pr[A_i]$ can only overestimate the probability of unions.

Coupon Collector Problem:

- There are n different baseball cards.

- Choose m cards at random with replacement.

1) what is the probability of failing to pick the k^{th} card.

A_k

$$Pr[A_k] = \left(\frac{n-1}{n}\right) \times \left(\frac{n-1}{n}\right) \times \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-1}{n}\right)$$

$$= \left(\frac{n-1}{n}\right)^m.$$

2) what is the probability of failing get at least one of the cards? [Find an upper bound]

$A_k =$ fail to pick the k^{th} card.

$$A = A_1 \cup A_2 \cdots \cup A_n$$

$$Pr[A_1 \cup A_2 \cdots \cup A_n] \leq \sum_{i=1}^{n} Pr[A_i]$$

$$= \sum_{i=1}^{n} \left(\frac{n-1}{n}\right)^m$$

$$= n \left(\frac{n-1}{n}\right)^m.$$