Random Variables

July, 22, 2020

1) Random variables

2) Probability Distribution

3) Multiple Random Variables

4) Expectation

Questions about outcomes...

- Experiment: roll two dice.

  Sample space: $\{(1,1), (1,2), \dots, (6,6)\} = \{1, \dots, 6\}^2$

  what's the summation?

- Experiment: flip 100 coins

  Sample space: $\{HH\dots H, THH\dots H, \dots, TTT\dots T\}$

  How many heads in 100 coins?

- Experiment: choose a random student in CS 70

  Sample space: $\{S_1, S_2, \dots, S_n\}$

  What is his/her midterm score?

- Experiment: hand back assignments to 3 students at random

  Sample space: $\{123, 132, 213, 231, 312, 321\}$

  How many students get back their own assignment?

In each scenario, the outcome is a number.

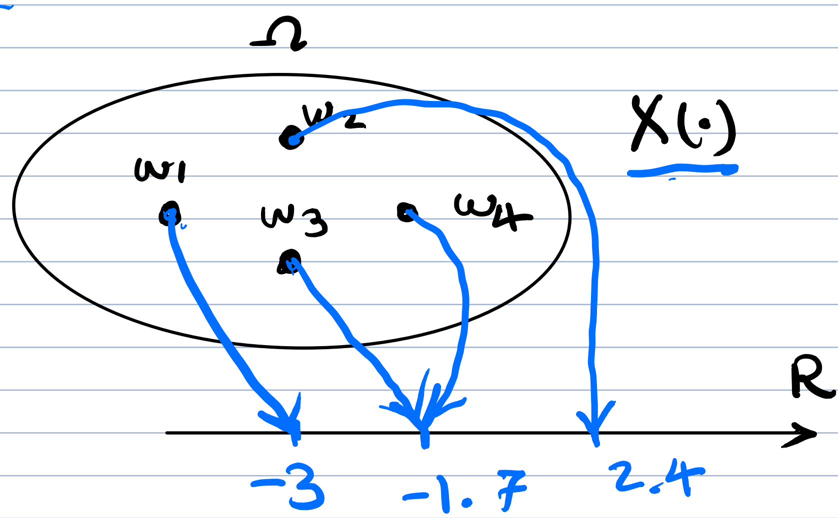
The number is a known function of outcomes.

1) Random Variables

A Random Variable, X, for an experiment with sample space Ω is a function $X: \Omega \to \mathbb{R}$

outcome

Thus, $X(\cdot)$ assigns? a real number $X(\omega)$ to each $\omega \in \Omega$



The function $X(\cdot)$ is defined on the outcomes Ω.

The function $X(\cdot)$ is not random, not a variable.

What varies at random (from experiment to experiment)?
The outcome.

Definitions:

(a) For $a \in \mathbb{R}$, one defines:

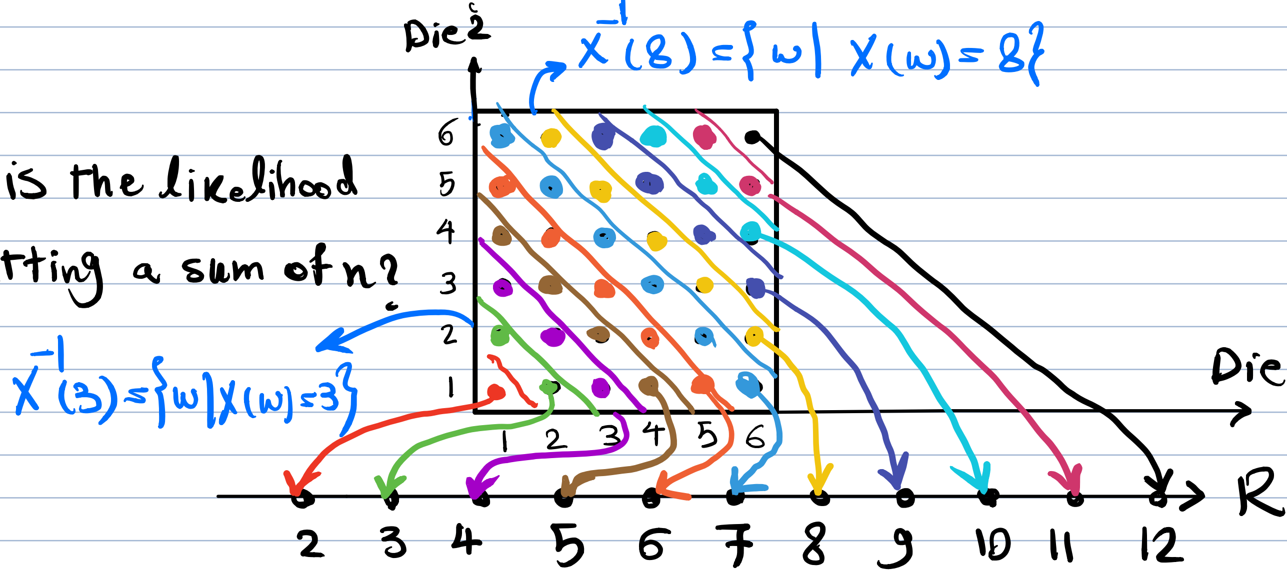
$$X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) = a\}$$

(b) The Probability that $X=a$ is defined as

$$Pr[X=a] = Pr[X^{-1}(a)] = \sum_{\omega:\, X(\omega)=a} Pr[\omega]$$

The summation of number on two rolled dice.

What is the likelihood of getting a sum of n?



$$\Pr[X=3] = \Pr[X^{-1}(3)] = \frac{2}{36}$$

$$\Pr[X=8] = \Pr[X^{-1}(8)] = \frac{5}{36}$$

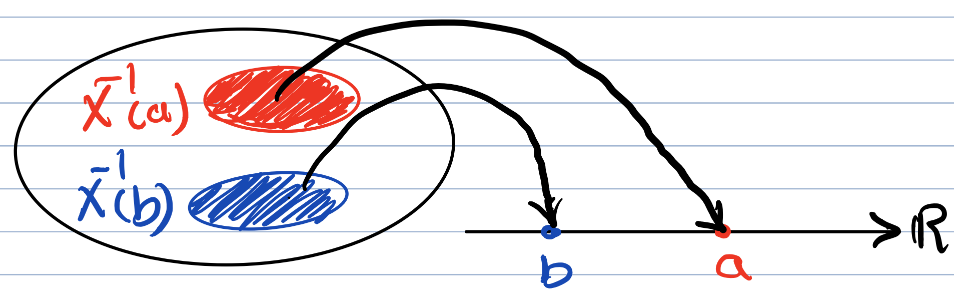
2) Probability Distribution:

The Probability of X taking on a value a.

Definition: The distribution of a random variable X, is

$$\{(a, \Pr[X=a]) ;\ a \in \mathcal{A}\}$$

where $\mathcal{A}$ is the range of X.



Examples:

Handing back assignments:

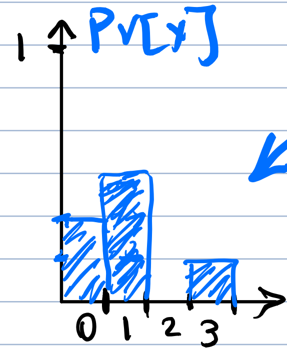
Hand back assignments to 3 students at random.

$$\Omega = \{\underbrace{123}_{\omega_1}, \underbrace{132}_{\omega_2}, 213, 231, 312, \underbrace{321}_{\omega_6}\} \Rightarrow 3! = 6$$

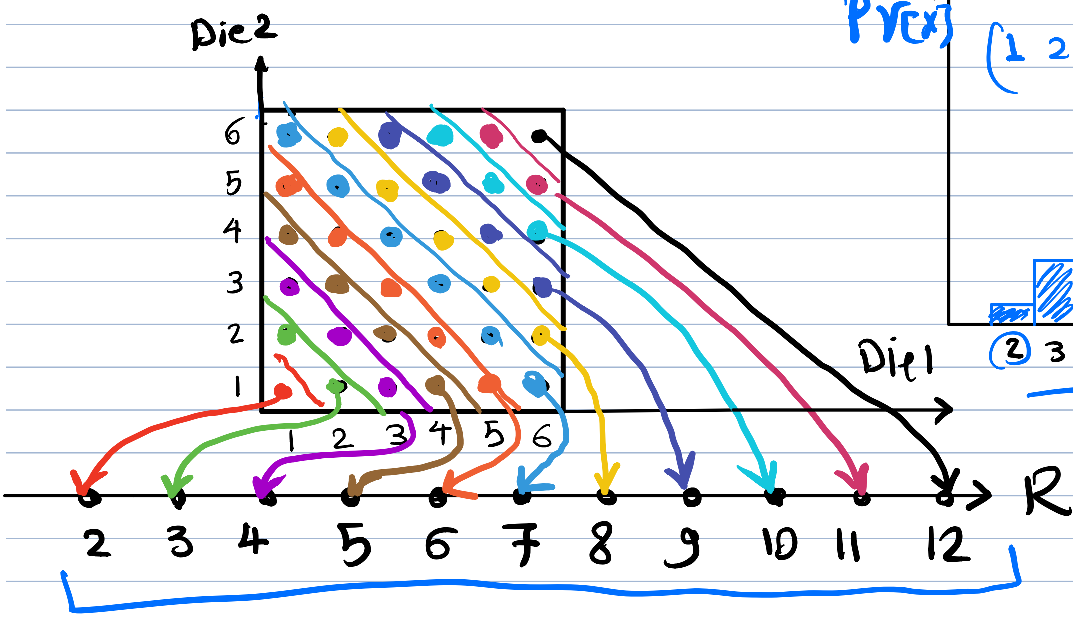
How many students get back their own assignments?

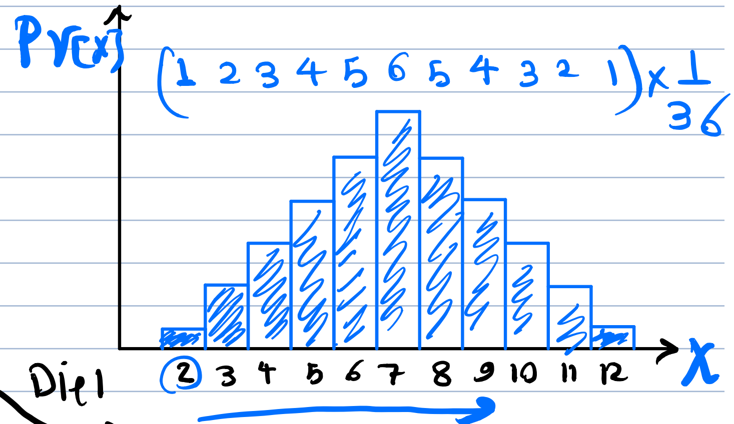
Random Variable: $X(\omega) = \{3, 1, 1, 0, 0, 1\}$

$$X = \begin{cases} 0 & Pr = \frac{2}{6} = \frac{1}{3} \\ 1 & Pr = \frac{3}{6} = \frac{1}{2} \\ 3 & Pr = \frac{1}{6} \end{cases}$$



The summation of number on two rolled dice





Named Distributions

Some distributions come up over and over again.

Bernoulli Distribution:

Flip a biased coin with heads probability p

Random variable: X takes heads (1) or tails (0)

X: a random variable that takes $\{0,1\}$

$$X = \begin{cases} 1 \\ 0 \end{cases}, \qquad \boxed{Pr[X=i] = \begin{cases} p & \text{if } i=1 \\ 1-p & \text{if } i=0 \end{cases}}$$

A Bernoulli random variable X is written as

$$X \sim \text{Bernoulli}(p).$$

Binomial Distribution:

Flip n biased coins with heads probability p.

Random variable: number of heads X

$$Pr[X=i], \quad \text{for } \boxed{i = 0, 1, \ldots, n}$$

How many sample points in events "$X=i$"?

i heads out of n coin flips $\Rightarrow$ $\boxed{\binom{n}{i}}$

What is the probability of w if w has i heads?

Pr. of heads p

Pr. of tails $1-p$

$$\star\ \Pr[w] = p^i \times (1-p)^{n-i}$$

Probability of "$X=i$" is sum of $\Pr[w]$, $w \in$ "$X=i$"

$$\Pr[X=i] = \Pr[X^{-1}(i)] = \sum_{w:\,X(w)=i} \Pr[w]$$

$$= \sum_{w:\,X(w)=i} p^i (1-p)^{n-i} = \binom{n}{i} p^i (1-p)^{n-i}.$$

$$\sum_{i=0}^{n} \Pr[X=i] = 1 \Rightarrow \boxed{\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1}$$

Binomial Theorem

Example:

Error channels

A Packet is corrupted with Probability p.

Send $n+2k$ Packets.

Probability of at most k corruptions

X: number of corruptions

$$\Pr[X \le k] = \sum_{i=0}^{k} \Pr[X=i]$$

$$= \sum_{i=0}^{k} \binom{n+2k}{i} p^i (1-p)^{n+2k-i}.$$

3) Multiple Random variables

One may be interested in multiple random variables.

- The concept of a distribution can then be extended for the combination of values for multiple variables.

Definition. Joint distribution.

The joint distribution for two discrete random variables X and Y is

$$\boxed{\{((a,b), \Pr[\underline{X=a, Y=b}]) : a \in \mathbb{A}, b \in \mathbb{B}\}}$$

where $\mathbb{A}$ is the set of all possible values taken by X and $\mathbb{B}$ is the set of all values take by Y.

Think of it as $\Pr[A \cap B]$ where $A: X=a$ and $B: Y=b$.

Then, what is $\Pr[X=a]$? marginal distribution

$\Pr[X=a]$ is determined by summing over all values for Y.

$$\Pr[X=a] = \sum_{b \in \mathbb{B}} \Pr[X=a, Y=b]$$

In the case with more random variables $X_1, X_2, \dots, X_n$ then, the joint distribution is

$$Pr[X_1=a_1, \dots, X_n=a_n], \quad a_i \in A_i$$

and $$\sum_{j=1}^{n} \sum_{a_j \in A_j} Pr[X_1=a_1, \dots, X_n=a_n] = 1$$

How to find $\underbrace{Pr[X_i=a_i]}$?

$$Pr[X_i=a_i] = \sum_{\substack{j=1 \\ j\neq i}}^{n} \sum_{a_j \in A_j} Pr[X_1=a_1, \dots, X_n=a_n]$$

Example:

Y \ X	1	2	3	4	5	6	7	Pr[y]
0	0.15	0	0	0	0	0.1	0.05	0.3
2	0	0.05	0.05	0	0	0	0	0.1
5	0	0	0	0.05	0.05	0	0	0.1
8	0.15	0	0	0	0	0	0.35	0.5
Pr[x]	0.3	0.05	0.05	0.05	0.05	0.1	0.4	1

- Definition: (Independence) Random variables X and Y are independent if the events $X=a$ and $Y=b$ are independent for all values a, b.

$$Pr[X=a, Y=b] = Pr[X=a]\,Pr[Y=b], \quad \forall\, a, b$$

Example: Indicators [very important]

. flip a coin n times

Define: I_i is the indicator $\boxed{\text{r.v.}}$ for the i^{th} coin flip.

$I_1, \dots, I_n$ are mutually independent.

This is known as independent and identically distributed (i.i.d.) set of random variables

Hence, $\{I_1, \dots, I_n\}$ is a set of i.i.d. indicator random variables.

- We extensively use indicators later

Difinition: combining random variables.

Let X, Y, Z be R.V. on Ω and $g: \mathbb{R}^3 \to \mathbb{R}$ a function.

Then $g(X, Y, Z)$ is the R.V. that assigns the value $g(X(\omega), Y(\omega), Z(\omega))$ to ω.

If $V = g(X, Y, Z)$ then $V(\omega) = g(X(\omega), Y(\omega), Z(\omega))$

Exampels: $X+Y$, XY, $(X+Y-Z)^3$

Conditional Probability for distributions:

$$\Pr[X=a \mid Y=b] \equiv \Pr_{X|Y}[a|b] = \frac{\Pr[X=a, Y=b]}{\Pr[Y=b]}.$$

4) Expectation:

- sometimes it is very hard to calculate the complete distribution of a r.v. .

- would like to summarize the distribution into a more compact, convenient form that is also easier to compute.

- The most widely used such form is the expectation (or mean or average) of the r.v.

Definition: The expected value of a random Varible X is

$$\underline{E[X] = \sum_{a \in A} a \times Pr[X=a]}$$

Theorem : $$E[X] = \sum_{w \in \Omega} X(w) \times Pr[w]$$

Proof: $$E[X] = \sum_{a} a \, \underline{Pr[X=a]}$$

$$= \sum_{a} a \sum_{w : X(w)=a} Pr[w]$$

$$= \sum_{a} \sum_{w : X(w)=a} a \, Pr[w]$$

$$= \sum_{a} \sum_{w : X(w)=a} X(w) \, Pr[w] = \underline{\sum_{w \in \Omega} X(w) Pr[w]}$$

Example: Roll a fair die

X: value on the die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$Pr[\omega] = \frac{1}{6}$

$$E[X] = \sum_a a\,Pr[X=a] = \sum_{a=1}^{6} \frac{a}{6} = 1\times\frac{1}{6} + 2\times\frac{1}{6} + \cdots + 6\times\frac{1}{6} = \frac{7}{2}.$$

Note: $E[X]$ doesn't have to be in the range of X.

- The expected value is not the value that you expect!

Example: Rolled two dice.

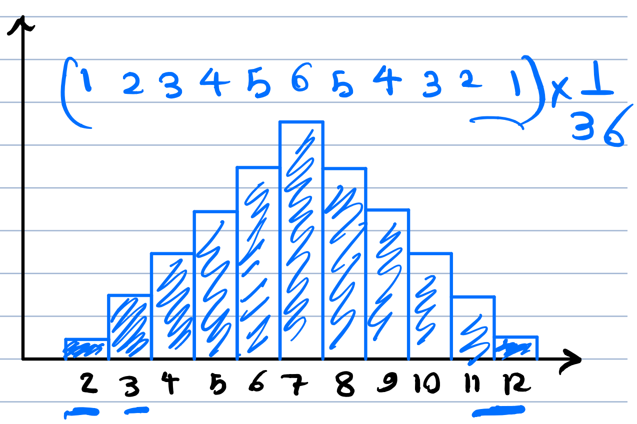
X: Summation of the rolled dice

$$E[X] = \sum_a a\,Pr[X=a]$$

$$= \sum_{a=2}^{12} a\,Pr[X=a]$$

$$= \cdots = 7$$

Not convenient!



Expectation of Binomial Distribution:

$$X \sim Bin(n, p)$$

$$Pr[X=i] = \binom{n}{i} p^i (1-p)^{n-i}$$

$$E[X] = \sum_a a \Pr[X=a] = \sum_i i \binom{n}{i} p^i (1-p)^{n-i}$$

not easy!

Linearity of Expectation:

Theorem: For any two random variables X and Y we have

$$E[X+Y] = E[X] + E[Y].$$

Also for any constant c,

$$E[cX] = cE[X]$$

In General: $E[c_1 X_1 + \cdots + c_n X_n] = c_1 E[X_1] + \cdots + c_n E[X_n]$

Proof: $E[\underbrace{c_1X_1+\cdots+c_nX_n}_{Z}] = \sum_{\omega} Z(\omega)\Pr[\omega]$

$= \sum_{\omega} [c_1X_1(\omega)+\cdots+c_nX_n(\omega)]\Pr[\omega]$

$= c_1 \underbrace{\sum_{\omega} X_1(\omega)\Pr[\omega]} + \cdots + c_n \underbrace{\sum_{\omega} X_n(\omega)\Pr[\omega]}$

$= c_1 E[X_1] + \cdots + c_n E[X_n]$.

- There is no assumption on r.v. $X_1, \ldots, X_n$
- r.v. $X_1, \ldots, X_n$ do not need to be independent!

Example: Rolled two dice.

X: Summation of the rolled dice

we can write:

$X = X_1 + X_2$

X_1: Number on die 1

X_2: number on die 2

$E[X] = E[X_1+X_2] \underset{\text{linearity}}{=} E[X_1]+E[X_2] = \frac{7}{2}+\frac{7}{2}$

$= 7$.

Expectation of Binomial Distribution:

Consider the biased coin flip example.

Define: I_i is an indicator r.v. for i^{th} flip being heads.

$$I_i = \begin{cases} 1 & i^{th} \text{ flip is heads} \quad Pr[I_i=1]=p \\ 0 & i^{th} \text{ flip is tails} \quad Pr[I_i=0]=1-p \end{cases}$$

Define: X = # of heads in n flips

$$X = I_1 + I_2 + \cdots + I_n = \sum_{i=1}^{n} I_i$$

we had $E[X] = \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$, not easy

$$E[X] = E\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} E[I_i] = \sum_{i=1}^{n} p = np$$

linearity

$$E[I_i] = \sum_{I \in \{0,1\}} I_i \, Pr[I_i] = 0 \times Pr[I_i=0] + 1 \times Pr[I_i=1] = 0 + p = p$$

$$\boxed{E[X] = np}$$