Markov Chains I

Lec.26

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Markov Chains: Fundamental Idea

We now wish to model sequences of random variables $\underline{X_0, X_1, X_2}, \ldots$. You can think of $\underbrace{X_n}$ as the state of a system at time n.

not iid., but the values that they take on all come from $\mathcal{X}$

We will be working on the setting where time is discrete, and each X_i can only take on a finite set of values. This finite set of values is denoted $\mathcal{X}$ and is called the state space.

Markov Property

Think of X_n be the present/current state, and X_{n+1} as the future state. The Markov Property is:

$$\Pr(X_{n+1} = j | X_n = i, \ldots, X_0 = i_0) = \Pr(X_{n+1} = j | X_n = i)$$

This property is not saying the future (X_{n+1}) is independent of the past ($X_0, \ldots X_{n-1}$). This property is saying that the past and future are *conditionally independent* given the present.

Note: time n is missing

We call $\Pr(X_{n+1} = j | X_n = i) = P(i, j)$ the transition probability from state i to state j.
In this class, we will only deal with time homogeneous Markov chains.

Transition Probability Matrix

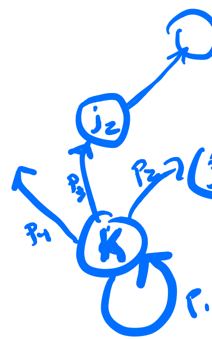
Let the state space $\mathcal{X}$ be $\{1, ..., k\}$. The transition probability matrix for a Markov chain P is a k by k matrix such that the entry in the ith row and j column is $P(i,j)$, and:

the prob. of going from state i to state j in 1 step

$$P(i,j) \geq 0 \quad \forall i,j \in \mathcal{X} \tag{1}$$

" the rows of P " sum to 1

$$\sum_{j=1}^{k} P(i,j) = 1 \quad \forall i \in \mathcal{X} \tag{2}$$



" the probability of going somewhere is 1 "

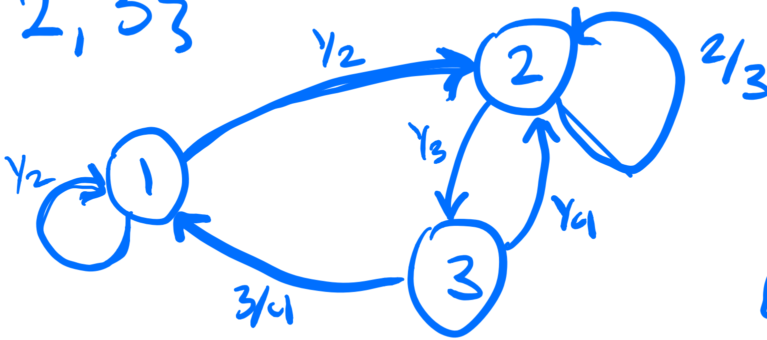
Note: P(i,i) is valid (you can go from a state back to the same state)

Markov Chain Example

$\mathcal{X} = \{1, 2, 3\}$

$X_0, X_1, X_2, \ldots, X_n \ldots$

$\pi_0(1) = 1$



Potential initial distributions

$\pi_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$\pi_0 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$

$\pi_0(1) = 1/3$

$\vdots$

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 \\ 3/4 & 1/4 & 0 \end{bmatrix}$$

i^{th} row j^{th} column is the probability of going from state i to state j in one timestep.

Ex: $\pi_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$$\pi_1 = \pi_0 P = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 \\ 3/4 & 1/4 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \end{bmatrix}$$

Distribution Over States

If there are k states, π_i is a $1 \times k$ row vector.

We use π_i to represent the distrution of our random variable X_i over the states in $\mathcal{X}$. The entries in π_i must be probabilities that sum up to 1.

π_0 is called our initial distribution. π_0, in conjunction with P and $\mathcal{X}$ fully specifies our Markov chain.

$$\pi_n(i) = \mathbb{P}(X_n = i), \text{ where } i \in \mathcal{X}$$

Moving in Time

Suppose at time n, X_n has distribution π_n. Then, by the Law of Total Probability,

$Pr(X_{n+1} = j \wedge X_n = i)$

$$\Pr(X_{n+1} = j) = \sum_i \Pr(X_{n+1} = j | X_n = i) \Pr(X_n = i) \tag{3}$$

$$= \sum_i P(i,j)\pi_n(i) \tag{4}$$

$= [\ 0\ ]$

$(1 \times K) \cdot (K \times K) = 1 \times K$

But $\sum_i P(i,j)\pi_n(i)$ is the jth entry of $\pi_n P$. So,

$\pi_1 = \pi_0 P$

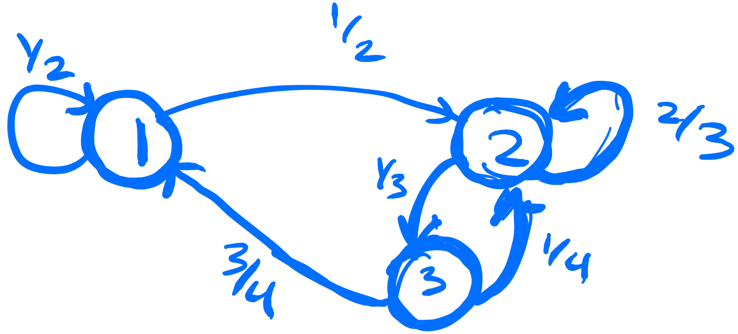
$$\pi_{n+1} = \pi_n P \tag{5}$$

$$\pi_{n+2} = \pi_{n+1} P = \pi_n P P = \pi_n P^2 \tag{6}$$

P^k is the probability transition matrix where the entry in the ith row jth column is the probability of going from state i to state j in k steps.

Hitting Time Example

What is the average number of steps it takes to reach state 1 starting at state 2?



Let $B(i)$ denote the average # of steps needed to reach state 1 starting from state i.

Then,

$$B(1) = 0$$

$$B(3) = 1 + \tfrac{3}{4}\cdot B(1) + \tfrac{1}{4}\cdot B(2)$$

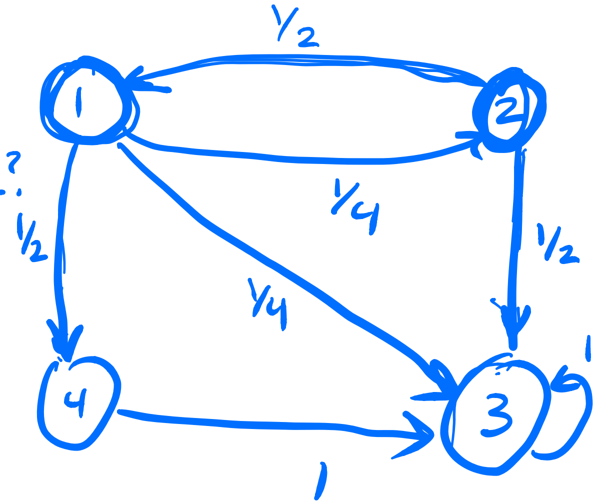
$$B(2) = 1 + \tfrac{2}{3}\cdot B(2) + \tfrac{1}{3}\cdot B(3)$$

3 eqns, 3 variables. Solve $\Rightarrow$

$$B(1) = 0$$
$$B(2) = \frac{16}{3}$$
$$B(3) = \frac{7}{3}$$

Probability of A before B Example

What is the probability of reaching state 3 before state 4, starting from state 1?



Let $\alpha(i)$ be the probability of reaching state 3 before state 4, starting from state i

$$P = \begin{bmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then, $\alpha(3) = 1$

$\alpha(4) = 0$

$\alpha(2) = \frac{1}{2}\cdot\alpha(1) + \frac{1}{2}\cdot\alpha(3)$

$\alpha(1) = \frac{1}{2}\alpha(4) + \frac{1}{4}\cdot\alpha(3) + \frac{1}{4}\alpha(2)$

4 eqn. 4 variables.

solve $\Rightarrow$

$\alpha(3) = 1$

$\alpha(4) = 0$

$\alpha(2) = \frac{10}{14}$

$\alpha(1) = \frac{3}{7}$

Invariant Distribution Definition

" steady state " distribution

" stationary " distribution.

A distribution π is *invariant* for the transition probability matrix P if it satisfies the following *balance equations*:

$$\pi = \pi P \tag{7}$$

Stationary Distribution Existence

Let P be the probability transition matrix for a Markov chain.

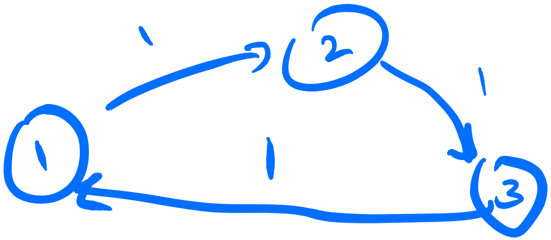
The rows of P add up to 1. Let $\mathbf{1}$ be a column vector of ones. This means $P\mathbf{1} = \mathbf{1} = 1 \cdot \mathbf{1}$.

This means P has a right eigenvector corresponding to eigenvalue 1. Since the right and left eigenvalues of a square matrix are the same, this means there exists some left eigenvector π such that $\pi P = 1 \cdot \pi$.

Note that this does not say anything about the uniqueness of the stationary distribution.

"stationary distr. is a left eigenvector of P
w/ eigenvalue 1"

Stationary Distribution Example



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\pi = [1/3 \quad 1/3 \quad 1/3]$

$\pi = \pi P$ balance eqns.

$$[\pi(1) \quad \pi(2) \quad \pi(3)] = [\pi(1) \; \pi(2) \; \pi(3)] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

lin dep. need 1 more eqn.

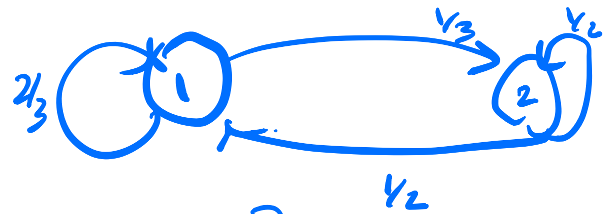
$$\begin{cases} \pi(1) = \pi(3) \\ \pi(2) = \pi(1) \\ \pi(3) = \pi(2) \end{cases}$$

replace an eqn. w/ $\pi(1) + \pi(2) + \pi(3) = 1$

$\Rightarrow \pi(1) = 1/3$

$\pi(2) = 1/3$

$\pi(3) = 1/3$



$$P = \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$$

$\pi = \pi P$

$$[\pi(1) \quad \pi(2)] = [\pi(1) \quad \pi(2)] \begin{bmatrix} 2/3 & 1/3 \\ 1/2 & 1/2 \end{bmatrix}$$

dep.

$$\begin{cases} \pi(1) = \pi(1)\, 2/3 + \pi(2)\, \tfrac{1}{2} \\ \pi(2) = \pi(1) \cdot 1/3 + \pi(2) \left(\tfrac{1}{2}\right) \end{cases}$$

replace an eqn w/ $\pi(1) + \pi(2) = 1$

$\Rightarrow \pi(1) = 3/5$

$\pi(2) = \frac{2}{5}$