Markov Chains II

Lec.27

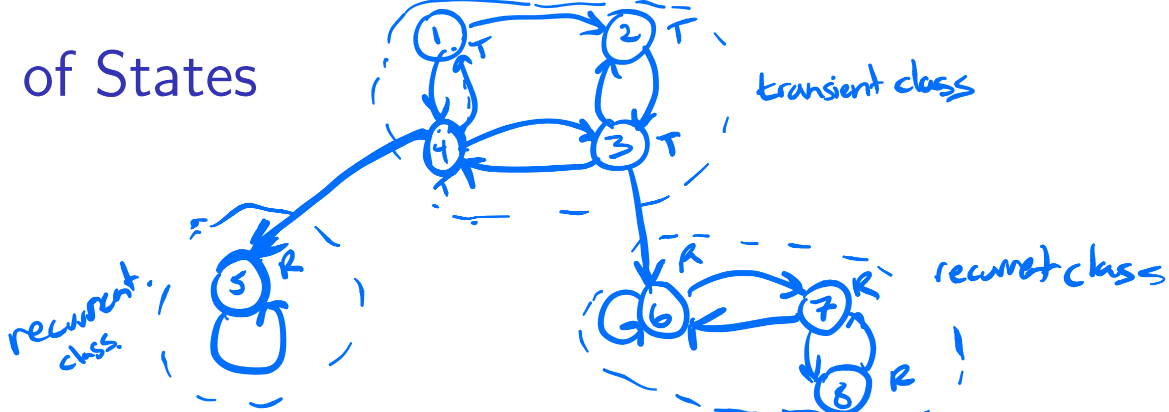
August 6, 2020

Invariant Distribution Recap

A distribution π is *invariant* for the transition probability matrix P if it satisfies the following *balance equations*:

$$\pi = \pi P \tag{1}$$

Classification of States

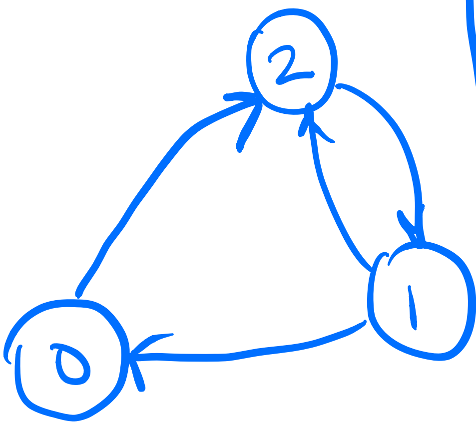


1. A state i is recurrent if starting from i, no matter what path we take, we can always return to i
2. A state i is transient if starting from i, there exists a path for which there is no way back to i
3. A class of states is a set of states where it is possible to get from any state to any other state

Irreducibility Definition

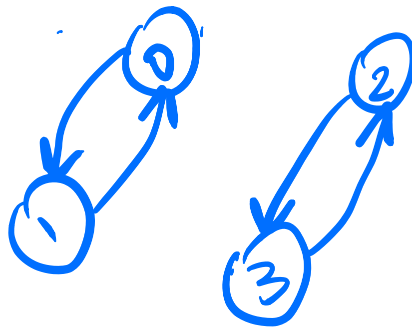
A Markov chain is irreducible if it can go from every state i to every other state j, possibly in multiple steps.

Irreducibility Example



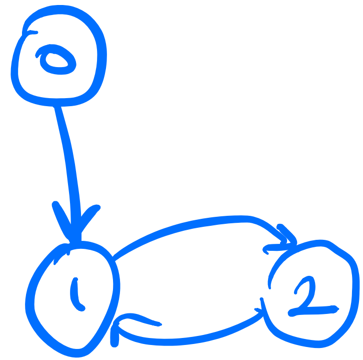
irreducible

bcz you can go from any state i to any state j



reducible.

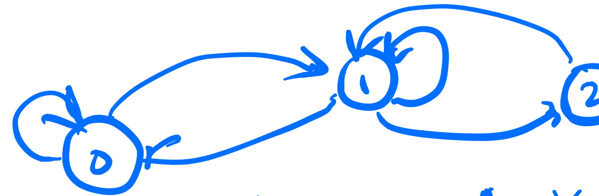
bcz you cannot go from 1 to 2 for example.



reducible

bcz you cannot go from 2 to 0 for example.

LLN for Markov Chains



$n=3$

$j=2 \to \frac{0}{4}$

example

$X_0=0,\ X_1=1,\ X_2=1,\ X_3=1 \ \ldots$

$j=1 \to \frac{3}{4} \to \pi(1)$ as $n\to\infty$

For an irreducible Markov Chain, we have that:

1. The chain has a *unique* invariant distribution (stationary) $\pi = [\pi(1) \ \ldots \ \pi(n)]$.
2. For each $j \in \mathcal{X}$,

indicator which is "on" (or 1) when $X_m = j$

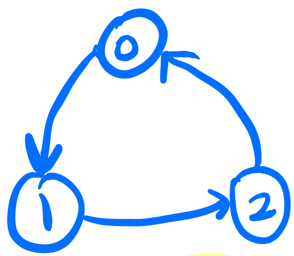
$|\mathcal{X}| = k$

$$\lim_{n\to\infty} \underbrace{\frac{\sum_{m=0}^{n-1} \mathbf{1}\{X_m = j\}}{n}} = \pi(j)$$

π is the unique stationary distribution.

$\uparrow$ $1 \times k$ row vector.

. This holds regardless of what particular π_0 we use.



$\pi_0 = [1 \ 0 \ 0]$

$\pi_1 = [0 \ 1 \ 0]$

$\pi_2 = [0 \ 0 \ 1]$

$\neq [1/3 \ \ 1/3 \ \ 1/3]$

$\uparrow$
π

$X_0=0,\ X_1=1,\ X_2=2,\ X_3=0,\ X_4=1,\ X_5=2 \ldots$

Periodicity Definition

Consider an irreducible Markov chain on $\mathcal{X}$ with transition probability matrix P. Define

$$d(i) := \text{g.c.d}\{n > 0 \mid P^n(i,i) = Pr[X_n = i | X_0 = i] > 0\}, i \in \mathcal{X}.$$

" length of all paths from a state to itself "

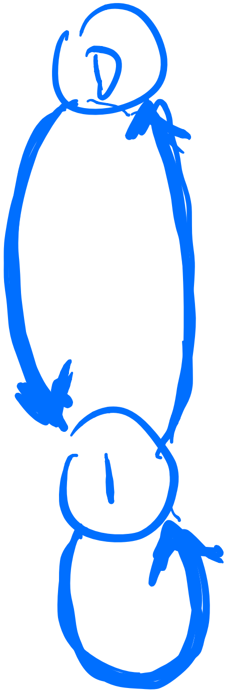
1. Then, $d(i)$ has the same value for all $i \in \mathcal{X}$. If that value is 1, the Markov chain is said to be *aperiodic*. Otherwise, it is said to be *periodic with period d*.
2. If the Markov chain is aperiodic, then

$$Pr[X_n = i] \to \pi(i), \forall i \in \mathcal{X}, \text{ as } n \to \infty. \tag{2}$$

where π is the unique invariant distribution. regardless of initial dist. π_0

For a given state i, the quantity $d(i)$ is the greatest common divisor of all the integers $n > 0$ so that the Markov chain can go from state i to state i in n steps.

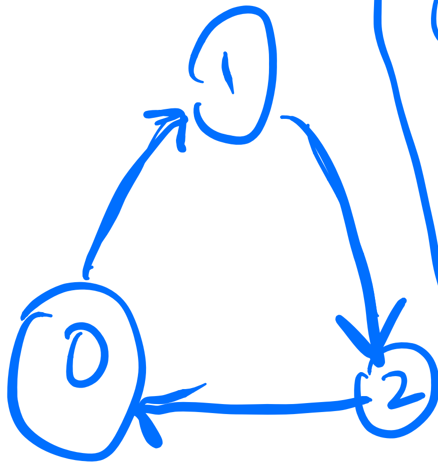
Periodicity Example



aperiodic $d(1)=1$

path lengths from 1 to 1

$\underbrace{\{1, 2, 3, \ldots\}}_{\text{gcd} \to 1}$

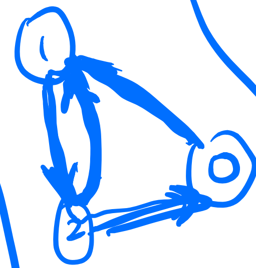


periodic.

$d(0)=3$

path lengths from 0 to 0

$\{3, 6, 9, \ldots\}$



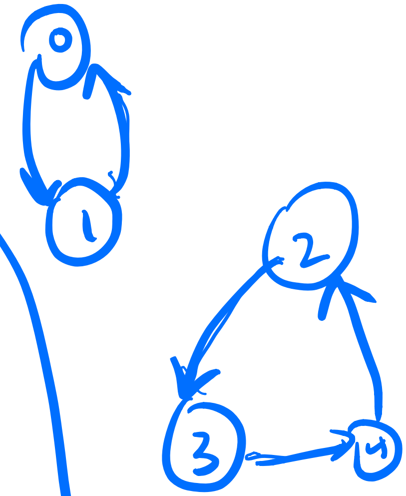
path lengths from 1 to 1

$\underbrace{\{2, 3, \ldots}_{\text{gcd} \to 1}$

$d(1)=1$

path lengths from 0 to 0

$\underbrace{\{3, 5, \ldots}_{\text{gcd } 1}$



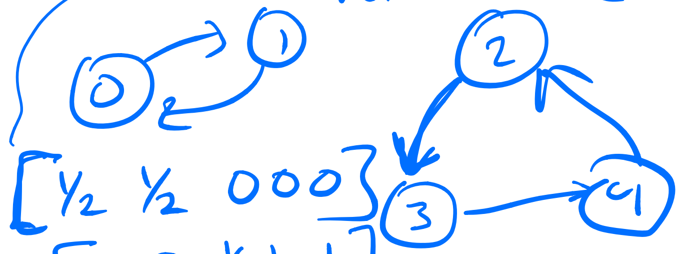
periodicity not defined

since it is not irreducible

Key Points

gcd → 3

not irreducible



solve to get stationary dist.

$$\begin{cases} \pi = \pi P \\ \sum_i \pi(i) = 1 \end{cases}$$

$\pi = [\frac{1}{2}\ \frac{1}{2}\ 0\ 0\ 0]$

$\pi = [0\ 0\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$

1. If a Markov chain is irreducible, it has a unique stationary distribution but does not necessarily converge to it
2. Periodicity is not defined for reducible Markov chains
3. If a Markov chain contains a self-loop, it is aperiodic. If there isn't a self-loop, it may or may not be aperiodic
4. If a Markov chain is irreducible and aperiodic, then it converges to a unique invariant distribution regardless of the initial distribution π_0

$\pi_0 \quad P \quad \rightarrow \pi_0 P \rightarrow \pi_0 P^2 \rightarrow \pi_0 P^3 \rightarrow$

then $\rightarrow \pi$